Flat Regular Models of Elliptic Schemes

Synopsis

Michael Szydlo

January, 1999

1 Synopsis

This thesis treats the problem of finding regular models for Elliptic Curves over general bases. One such model of an Elliptic Curve over a base that is a DVR with perfect residue field, or more generally a Dedekind domain, is the Néron model [NÉ]. One computes the reduction types of a Néron model directly with a Weierstrass equations by using Tate's Algorithm [TA]. So, the Néron model is an example of a regular model of an Elliptic curve over a one dimensional base. In a paper by Miranda [MIR], regular models of elliptic curves over two dimensional smooth surfaces over a field of characteristic zero were constructed.

I am interested in combining and extending these results to construct good models over a relatively general base scheme. We will assume the base scheme to be Noetherian, n dimensional, regular, integral and separated. I am particularly interested bases which may be high dimensional and of mixed characteristic.

I present here a self contained overview of the results, and a guide to where the results are proved in the thesis.

For simplicity, I assume that 1/6 is in all of the local rings of the base.

Let us define what type of models we are interested in constructing.

Definition 1.1 (Flat Resolution)
Let B be a regular Noetherian n - dimensional integral separated scheme. Let $X \to B$ be an elliptic subscheme of P^2/B defined locally by Weierstrass Equations. Suppose there exists a blow up $B' \to B$ defining the base change

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ B' & \to & B \end{array}$$

and a second Weierstrass elliptic scheme X'' birational to X' over B' with X'' regular, projective and flat over B'. Then the scheme $X'' \to B'$ is called a <u>*Flat Resolution*</u> *of $X \to B$.*

The notion of a flat resolution is a delicate one. For example, flat resolution is not functorial with respect to base change.

Now assume that the base B has no points of residue characteristic 2 or 3. Let B be, in fact, $Spec(O)$ where O is a regular local ring and let X be defined by the Weierstrass equation

$$y^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{1}$$

where $a_i \epsilon O$. Define the quantities c_4 and d as in [SIL 1].

As elements in O, both c_4 and d define divisors on B.

We make the important assumption that d has normal crossings on B. This means that locally

$$d = ut_1^{a_1}t_2^{a_2}....t_k^{a_k} \tag{2}$$

where the $\{t_i\}$ define linearly independent vectors in the cotangents space.

Furthermore assume that there is a morphism

$$J : B \to P^1(B) \tag{3}$$

which extends the rational map

$$p \mapsto (c_4^3, d) \tag{4}$$

to a regular map. We call this the J morphism. Our main theorem is

Theorem 1.2 (J Morphism Implies Flat Resolution)
Let B be a regular Noetherian n-dimensional integral separated scheme, and let $X \to B$ be an elliptic subscheme of P^2/B defined locally by Weierstrass Equations. Suppose the discriminant divisor has normal crossings, and there exists a J morphism

$$J : B \to P^1(B)$$

extending the j invariant for non-singular elliptic curves. Then $X \to B$ admits a flat resolution $X'' \to B'$.

We now briefly discuss the assumptions that the discriminant has normal crossings and that there is a J morphism.

Definition 1.3
Let B be a regular Noetherian n-dimensional integral separated scheme. Let D be a divisor on B. Suppose there exists a blow up $f : B' \to B$, (such that B' is also regular) such that $f^*(D)$ has normal crossings in B'. Then we say that the divisor D may be resolved in B.

Corollary 1.4
Let B be a regular Noetherian n-dimensional integral separated scheme on which 6 is a unit. Assume that divisors may be resolved in B. Then any Weierstrass elliptic scheme over B admits a flat resolution.

To prove this, we blow up the base so that d has normal crossings, and secondly we blow up the base at points where the map $p \mapsto (c_4^3, d)$ does not extend to a regular map.

Corollary 1.5
Let B be a regular Noetherian n-dimensional integral separated K-scheme where K is a field of characteristic 0. Then any Weierstrass elliptic scheme over B admits a flat resolution.

Before attacking the case where B may be of any dimension, we consider the simpler case where $B = Spec(R)$ where R is a DVR. This case is well known, and the regular model of theorem 1.2 is just the Néron model, (provided the residue field is perfect). Kodaira's symbols for the special fiber are I_0, I_n, II, III, IV, I_0^*, I_n^*, IV^*, III^*, II^*. Furthermore an algorithm of Tate constructs the Néron model [TA]. The reduction types and Tate's algorithm may also be found in [SIL 2].

Thus to each Elliptic curve over a fraction field of a DVR one associates the above type that Tate's algorithm produces.

I prove the following version, which does not require that k has perfect residue field.

Theorem 1.6 (Extension to Tate's Algorithm)
Let R be a DVR with maximal ideal p, a uniformizing element π, fraction field

K, and residue field k. Let E/K be an elliptic curve given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{5}$$

Then there exists a regular proper flat scheme C/R, with generic fiber E/K. If k is not of characteristic 2 or 3, the special fiber of this model one of Kodaira's types.

Thesis Reference Sections 3-6 prove this. The version where k can have residue characteristic 2 or 3 is Theorem 6.1 and Theorem 7.1. Special fibers not appearing on Kodaira's list are described in Theorem 7.2.

It is well known that elliptic curves with reduction types on Kodaira's list have specific values of j. Specifically, types II, IV, IV^*, and II^* have $j = 0$, types III, and III^* have $j = 1728$, and types I_n, and I_n^* have $j = \infty$.

We now consider base schemes $Spec(O)$ where O is a local ring. Let p be the closed point of $Spec(O)$.

Theorem 1.7 (Fibers of J)
Given any Weierstrass equation

$$y^2 = x^3 + a_2x^2 + a_4x + a_6. \tag{6}$$

with coefficients in O such that

$$d = ut_1^{a_1}t_2^{a_2}....t_k^{a_k} \tag{7}$$

(where u is a unit in the local ring , the a_i are integers, and the elements $t_i \epsilon O$ define linearly independent vectors in the cotangent space m/m^2), and there is a J morphism on Spec(O),

then there exists an O translation $x = x' + \alpha$ with $\alpha \epsilon O$ such that the translated Weierstrass equation

$$y'^2 = x'^3 + a_2'x'^2 + a_4'x' + a_6' \tag{8}$$

satisfies the following conditions for special values of $j(p)$.$*_1$

If $j(p) = 0$,

$$a_6' = ut_1^{a_1/2}t_2^{a_2/2}....t_k^{a_k/2}. \tag{9}$$

for some unit u.

If $j(p) = 1728$*,*

$$a_4' = ut_1^{a_1/3} t_2^{a_2/3} \ldots . t_k^{a_k/3}. \tag{10}$$

for some unit u.

If $j(p) = \infty$*,*

$$a_2' = ut_1^{b_1} t_2^{b_2} \ldots . t_k^{b_k} \tag{11}$$

for some unit u, where $b_i = 0$ *or* $b_i = 1$*, and*

$$a_4' = vt_1^{c_1} t_2^{c_2} \ldots . t_k^{c_k} \tag{12}$$

for some element v where $c_i \geq 2b_i$*, and*

$$a_6' = vt_1^{e_1} t_2^{e_2} \ldots . t_k^{e_k} \tag{13}$$

for some element v where $c_i \geq 3b_i$*.*

If $j(p) \neq \infty, 0, 1728$*, the* a_i *of equation 8 are all 0 or 6.*

$*_1$ A stronger condition is proved in the thesis. We show that the Weierstrass equation can be translated to multiple chart form. This is defined in Section 11 prop 11.1. To read this section, one requires the charts of Section 4, and the discussion in Section 8, particularly Proposition 8.12.

Further Thesis Reference The above theorem is proved in various sections of the thesis. Definition 10.2 defines groups of reduction types, and Proposition 15.3 shows that a J morphism will limit collision to types in a given group. Finally, Proposition 10.11 shows that the a_i have the form in the theorem above.

A Weierstrass equation such as 8 with coefficients in local ring O_p also defines elliptic curves over DVRs as follows. If $d = ut_1^{a_1} t_2^{a_2} \ldots . t_k^{a_k}$, we define $\underline{E_i}$ to be the elliptic curve defined by tensoring the Weierstrass equations with the DVR

$$R_i = \{f/g\} \quad f \epsilon O, f \epsilon O - \{(t_i)\}.$$

We say that the reduction types of the curves E_i collide at p.

Given a Weierstrass elliptic scheme over a local ring O, if we blow up the base $B \to O$ at a reduced closed subscheme of the discriminant divisor, and consider the scheme over B obtained by pullback $*_2$, the exceptional divisor in B, may be part of the new discriminant locus. If so, the reduction type over this component is determined by the following theorem.

Theorem 1.8 (Exceptional Divisor Determined)
Given any Weierstrass equation

$$y^2 = x^3 + a_2x^2 + a_4x + a_6. \tag{14}$$

with coefficients in O such that three conditions hold:

1. $$d = ut_1^{a_1}t_2^{a_2}....t_k^{a_k} \tag{15}$$
   (where u is a unit in the local ring , the a_i are integers, and the elements $t_i \epsilon O$ define linearly independent vectors in the cotangent space m/m^2).
2. *There is a J morphism $J : Spec(O) \to P^1(O)$.*
3. *The translation of 1.7 is the trivial translation $x' = x$.*

then if $B \to O$ is the blowing up at the ideal

$$(t_{\lambda_1}, ...t_{\lambda_r}) \tag{16}$$

where the t_{λ_1} are a subset of the $\{t_i\}$, and $X \to B$ is the scheme obtained by pulling back equation 14 to B, $*_2$,

then for any local ring $O_{p'}$ of B (in which $s \epsilon O_p$ defines the exceptional divisor on B)

the elliptic curve E_s obtained by tensoring 8 with the DVR

$$R_s = \{f/g\} \quad f \epsilon O_{p'}, f \epsilon O_{p'} - \{(s)\},$$

satisfies the following conditions depending on $j(p)$.

If $j(p) = 0$, then $a_6 = us^p$, where u is a unit in R_s and p is the integer $\sum a_i/2$ modulo 6, where the integers $a_i/2$ are the exponents of the t_{λ_j} appearing in equation 9.

If $j(p) = 1728$, then $a_4 = us^p$, where u is a unit in R_s and p is the integer $\sum a_i/3$ modulo 4, where the integers $a_i/3$ are the exponents of the t_{λ_j} appearing in equation 10.

If $j(p) = \infty$, then $a_2 = us^p$, where u is a unit in R_s and p is the integer $\sum b_i$ modulo 2, where the integers b_i are the exponents of the t_{λ_j} appearing in equation 11.

If $j(p) \neq 0, 1728, \infty$, then $d = us^p$, where u is a unit in R_s and p is the integer zero or 6.

For all values of j, $d = us^p$, where u is a unit in R_s and p is the integer $\sum a_i$ modulo 12, where the integers a_i are the exponents of the t_{λ_j} appearing in equation 15.

$*_2$ After blowing up the base an pulling back the scheme X to the new base we replace our scheme with one that is birational to X as in section 8.6. For example, if $t^6 \mid a_6$, we replace the Weierstrass equation $Y^2 = X^3 + a_6$ with $Y^2 = X^3 + \frac{a_6}{t^6}$.

Further Thesis Reference It is the special form of the Weierstrass equations which allows us to compute the effect of the blow ups in coordinates. Section 9.4 describes the powers of s in the coefficients a_i and d, and Proposition Prop 13.3 specifies the above 'collision arithmetic' mod 2, 4, or 6.

Corollary 1.9 (Exceptional Divisor Type is Known)
Given a Weierstrass equation with coefficients in a regular local ring, the reduction type of E_s/R_s may be computed just from the reduction types of E_i/R_i.

Thesis Reference This is corollary 13.6. To prove this, we use the value of J to limit the reduction type to one of 6, 4, or 2 types and the value of s in theorem 1.8 to pinpoint which it is.

Thus the reduction type of the exceptional divisor depends only on the reduction types E_i/R_i, and the ideal at which we blow up. For example, suppose a type III and a type I_0^* collide at a point, and we blow up at the ideal (t_1, t_2) where t_1, (resp., t_2) defines the component of the reduced discriminant associated to the type III (resp., I_0^*). The elliptic curve E_s/R_s has reduction type III^*.

Suppose now B is a base scheme that may not be local. We may apply theorems 1.7 and 1.8 to each local ring of the base. In particular, if we blow up the base B, at subschemes defined by the intersection of two or more components of the reduced discriminant divisor, we may predict the exceptional divisor type locally (theorem 1.8 of this section), and further limit collisions of reduction types.

Theorem 1.10 (Limited Collisions)
There exists a series of blow ups $B' \to B$ $*_3$ such that for any local ring O of B', the scheme obtained by pulling back equation 14 to O, $*_2$, satisfies the following conditions depending on $j(p)$ $*_4$.

If $j(p) = 0$, for at most two i, the elliptic curve E_i / R_i has reduction type different than I_0. If there are exactly two such i the two types are one of the following pairs $(II, I_0^*), (II, IV^*), (IV, I_0^*)$.

If $j(p) = 1728$, for at most two i, the elliptic curve E_i / R_i has reduction type different than I_0. If there are exactly two such i the two types are type III, and I_0^*.

If $j(p) \neq 1728, 0, \infty$, for at most one i, the elliptic curve E_i / R_i has reduction type different than I_0. If there is one such i the types must be I_0^*.

If $j(p) = \infty$, for at most one i, is the elliptic curve E_i / R_i has reduction type different I_n^*. Among the i such that the reduction type is I_{m_i}, at most one m_i is odd.

$*_3$ These blow ups are specified in section 14.

$*_4$ These rigid conditions are defined in Definition 14.3, and in section 14 in general. (See the definition of a Limited Weierstrass elliptic scheme).

Further Thesis Reference Proposition 14.4 limits the collisions when $j \neq \infty$.

Proposition 14.5 limits the collisions when $j = \infty$.

With these extreme conditions on the Weierstrass equations that locally define the elliptic scheme, we are able to desingularize $X \to B$.

Theorem 1.11 (Follow Tate to Desingularize)
Suppose $X \to B$ is an elliptic scheme for which the blow ups of 1.10 have already been performed. Then for there is a series of blow ups $*_5$ $*_6$

$$X'' \to ... \to X \tag{17}$$

such that the scheme ,X", is regular.

In fact, for each component of the discriminant divisor there may be some blow ups required. The series 17 is the compositum of these blow ups.

$*_5$ Section 16.3 shows how to define these blow ups, for a fixed component of the discriminant locus. Tate's algorithm actually specifies these blow ups.

$*_6$ For the order of these blow ups, we first order the components of the discriminant locus as in section 16.3.

One checks regularity, by examining the cotangent space m/m^2 of EVERY point of X''.

Further Thesis Reference We discuss the fibers off the discriminant divisor, or on a smooth point of the discriminant divisor in section 16.4.

We compute the fibers that belong to more than one component of the discriminant divisor in section 16.6 and in section 16.8.

By analyzing the special fibers individually, one checks that they are all one dimensional. This shows that X'' is flat over B. $*_7$.

$*_7$ This is discussed in section 16.3.3.

Corollary 1.12 (Special Fibers)
In the model X'' constructed, each special fiber is an elliptic curve, or a

collection of rational P^1's of specified multiplicity intersecting transversally $*_8$, *in one of the following configurations:*

$$1-2-3$$

$$1-2-3-2$$

$$1-2-3-2-1$$

$$1-2-3-4-2$$

$$I_0, II, III, IV, I_0^*, IV^*, III^*, II^*$$

$$I_n^+$$

In this corrollary, a chain $1-2-3$ means that a rational curve of multiplicity 1, intersects a rational curve of multiplicity 2, which in turn intersects a rational curve of multiplicity 3. The configurations in the fifth line are the standard Kodaira reduction types. The I_n^+ configuration consists of two multiplicity one components connected to a chain of $\frac{n-1}{2}$ multiplicity two components.

$*_8$ This means that the reduced special fiber is locally defined by equations $f = ut_1^{a_1}....t_k^{a_k}$ where the t_i define linearly independent vectors in the cotangent space.

Further Thesis Reference To strengthen the above theorem we can specify exactly which configuration the fiber has. See theorem 16.1.

For an involved chart which specifies the special fiber based on reduction types 'involved in the collision' (even at points of residue char 2 or 3) see Corollary 17.4.

2 References

References

[COR-SIL] Cornell, G, and Silverman, J., Arithmetic Geometry, Springer-Verlag, 1986.

[EIS-HAR] Eisenbud and Harris,J., Schemes, the Modern Language of Algebraic Geometry, Springer-Verlag, 1992.

[HAR] Harris,J., Algebraic Geometry, *Graduate Texts in Mathematics 133,*

[HART] Hartshorne, R., Algebraic Geometry, *Graduate Texts in Mathematics 52,*, Springer-Verlag, 1977.

[KOD] Kodaira,K,On Compact Analytic Surfaces II, Ann Math., 1963.

[MAT] Matsumura, H., Commutative Algebra, Benjamen and Cummings, 1980.

[MIR] Miranda, Regular Models of Elliptic Threefolds.

[NÉ] Néron, A., Modeles minimaux des varietietes abeliennes sur les corps locaux et globaux, *IHES Publ Math 21,* 1964, 361-482.

[OG] Ogg, A., Elliptic Curves and Wild Ramification *Am. J. of Math 89,*, 1967, 1-21.

[SIL 1] Silverman, J., The Arithemetic of Elliptic Curves *Graduate Texts in Mathematics 106,* . Springer-Verlag, 1991.

[SIL 2] Silverman, J., Advanced Topics in The Arithemetic of Elliptic Curves *Graduate Texts in Mathematics 151,* . Springer-Verlag, 1996.

[TA] Tate, J., Algorithm for determining the type of a singular fiber in an elliptic pencil, *Modular Functions of one Variable IV,* Lecture Notes in Math 476. Springer-Verlag, 1975, 33-52.