Flat Regular Models of Elliptic Schemes

Michael Szydlo

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1 Introduction

This thesis treats the problem of finding regular models for Elliptic Curves over general bases. One such model of an Elliptic Curve over a base that is a DVR with perfect residue field, or more generally a Dedekend domain, is the Néron model [NÉ]. We compute the reduction types of a Néron model directly with a Weierstrass equations by using Tate's Algorithm [TA]. So, the Néron model is an example of a regular model of an Elliptic curve over a one dimensional base. In a paper by Miranda [MIR], regular models of elliptic curves over two dimensional smooth surfaces over a field of characteristic zero were constructed.

I am interested in combining and extending these results to construct good models over a relatively general base scheme. We will assume the base scheme to be Noetherian, n dimensional, regular, integral and separated. I am particularly interested bases which may be high dimensional and of mixed characteristic.

In my original research, I made the assumption that 1/6 was in all of the local rings of the base, but here I present a coherent exposition without this limitation. As expected, new phenomena occur when the base contains points of residue characteristic two or three. The goal of this paper is to give criteria for the constructability of a regular model of the elliptic scheme, algorithmically construct it when possible, and to describe the fibers of the model.

Let us define what type of models we are interested in constructing.

Definition 1.1 (Flat Resolution)
Let B be a regular Noetherian n - dimensional integral separated scheme. Let $X \to B$ be an elliptic subscheme of $P^2(B)$ defined locally by Weierstrass Equations. Suppose there exists a blow up $B' \to B$ defining the base change

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ B' & \to & B \end{array}$$

and a Weierstrass elliptic scheme X'' birational to X' over B' with X'' reg-

ular, minimal, and projective and flat over B' Then the scheme $X'' \to B'$ is called a <u>Flat Resolution</u> of $X \to B$.

The notion of a flat resolution is a delicate one. For example, flat resolution is not functorial with respect to base change.

By an elliptic scheme over B, we mean a B-scheme whose generic fiber is an elliptic curve. See section 8 for the precise definition of this. We will define the notion of a *pre-settled and settled elliptic scheme* in Sections 10.8 and 12.4, and a J morphism in section 15. We provide criteria for an elliptic scheme to have a flat resolution. In particular in Section 16 We will prove

Theorem 1.2 (Settled Implies Flat Resolution)
Let B be a regular noetherian n-dimensional integral separated scheme, Let $X \to B$ be an elliptic subscheme of $P^2(B)$ defined locally by Weierstrass Equations. Suppose the discriminant divisor has normal crossings and that X is pre-settled and settled over B. Then $X \to B$ admits a flat resolution $X'' \to B'$.

If we assume that the base B has no points of residue characteristic 2 or 3, the J morphism will be sufficient for the elliptic scheme to have a flat resolution.

Theorem 1.3 (J Morphism Implies Flat Resolution)
Let B be a regular Noetherian n-dimensional integral separated scheme, Let $X \to B$ be an elliptic subscheme of $P^2(B)$ defined locally by Weierstrass Equations. Suppose the discriminant divisor has normal crossings, and there exists a J morphism

$$J : B \to P^1(B)$$

extending the j invariant for non-singular elliptic curves. Then $X \to B$ admits a flat resolution $X'' \to B'$.

This simple criteria for the existence of flat resolution over bases with no points of characteristic 2 or 3, leads me to conjecture the existence of flat resolution in general. Furthermore, through computer simulations that I have run, I have some evidence that it is always possible to reduce to a settled elliptic scheme.

This thesis actually shows much more than theorems 1.2 and 1.3. In the construction of the flat resolution we also obtain a precise description of the fibers of the morphism $X'' \to B$. The results of this finer analysis appear in section 6 and 16.

In particular, we describe geometrically the fibers over all closed points of B'. We will define a discriminant subscheme of the base B in Section 8. The fibers over closed points not in this subscheme are elliptic curves. The fibers of X'' over smooth points of the reduced discriminant locus of characteristic not equal to two or three are the special fibers on Kodaira's list of reduction types [KOD], and the fibers above points belonging to two or more components of the reduced discriminant locus are called collisions, (defined in section 10), and will be described geometrically.

Fibers over smooth points of the reduced discriminant locus of residue characteristic 2 and 3 may be new fiber types not on Kodaira's list. These new fiber types will be described in section 6. These new types as well as the standard ones may also collide if the reduced discriminant locus has multiple components. These collision fibers are also computed in section 16.

The construction of the regular model is sufficiently algorithmic to allow all fibers to be computed by using the extension of Tate's Algorithm and by consulting Chart 310 in section 17 to compute the collision fiber types.

2 Weierstrass Equations

If we were fortunate enough to have $1/6$ in the local rings of B, we could reduce our Weierstrass equations to a simpler form. In the case that the local ring does not have residue char 2, we can still complete the square and eliminate a_1 and a_3 to obtain

$$y^2 = x^3 + a_2x^2 + a_4x + a_6. \tag{1}$$

But when the local ring does have residue char 2 we must deal with the more general

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{2}$$

We are going to start our analysis when the base is a DVR, but later we will consider Weierstrass equations over higher dimensional local rings.

Explicitly we will study subschemes of $P^2(B)$ given by

$$Spec(R[x, y]/(f)) \tag{3}$$

where $B = Spec(R)$ is an affine base and f is the polynomial defined by equation 2.

3 A New Tate's Algorithm

3.1 Motivation

We will need a new Tate's Algorithm in order to pay special attention to translations and extraction of square and cube roots in the residue field.

Given a simplified form of the Weierstrass equation

$$y^2 = x^3 + a_4x + a_6. \tag{4}$$

Tate's algorithm easy to implement because no series of translations are needed. In general Tate's Algorithm modifies a Weierstrass equation via translations at several steps.

Definition 3.1 (R translation)
Let u, r, s, t be elements of the DVR R. An R translation of a Weierstrass equation is given by

$$x = u^2x' + r \tag{5}$$

$$y = u^3y' + u^2sx' + t \tag{6}$$

The values of u, r, s, t are specified in Tate's Algorithm, and require a square or cube root in the residue field [TA].

In this paper we are not able to assume that the residue field κ is perfect, and some of these translations may not be possible. In particular, if κ is not perfect an irreducible polynomial in $\kappa[x]$ may have double or triple roots. In this case the roots only exist in a nonseparable extension of the residue field.

Since we are unwilling to perform a finite base extension, we must treat these cases as part of the algorithm. The first step of the analysis is to construct a Tate's algorithm in cases where the residue field may not be perfect.

3.2 Elements of the Algorithm

We begin with a Weierstrass equation over a DVR, and would like to produce a regular proper minimal model for it. If the residue characteristic is not 2 or 3, we may use the original Tate's algorithm. Here we treat the cases of characteristic 2 and 3, and also carefully review the other cases remarking that at no step in the argument need we divide by 2 or 3.

The reason to avoid this division is that DVRs of residue characteristic 0 also arise as localizations of local rings of residue characteristic 2 or 3. When working with higher dimensional local rings O we will need to lift elements from localizations of O back up to O. Since none of the steps of the extended Tate's algorithm will require a division by 2 or 3, the fact that $\frac{1}{6}$ may not be in O presents no problem.

The essence of Tate's algorithm, in all residue characteristics, is to examine the valuations of the a_i's, and to perform translations on x and y to produce new a_i's of a special form. This form uniquely identifies the reduction type of the regular model.

The translations will be explicitly defined in sections 4 and 5, but in short they just translate the singular point on the special fiber to (0,0), and secondly assure that any multiple roots of the following polynomials are translated to zero.

$$Y^2 + a_1XY - a_2X^2 \tag{7}$$

$$Y^2 + \frac{a_3}{\pi}Y - \frac{a_6}{\pi^2} \tag{8}$$

$$X^3 + \frac{a_2}{\pi}X^2 + \frac{a_4}{\pi^2}X + \frac{a_6}{\pi^3} \tag{9}$$

$$Y^2 + \frac{a_3}{\pi^2}Y - \frac{a_6}{\pi^4} \tag{10}$$

There are other polynomials to consider in the I_n , I_n^*, K_n, and T_n families that we will later define.

3.3 Method of Proof

The proof of the extension to Tate's algorithm will be divided into cases depending on the characteristic of the residue field. The three cases will be: characteristic 2, characteristic 3, all other characteristics.

For each case I will define *Types* of the Weierstrass equation depending on the valuations of the a_i and other criteria also depending on the a_i. These types are defined in section 4, and will be summarized in various charts and subcharts. Not every Weierstrass equation will have a type, but in section 5 we will prove that every Weierstrass equations can be translated to one of these desired special forms. In section 6 we will use that special form to produce the regular projective minimal model of the elliptic curve. This will be done by a series of blow ups and then checking regularity.

The use of charts and chart types is just a convenient method of organizing the data contained in the Weierstrass coefficients. The blow-ups actually used to construct the model could actually be defined in a translation independent way.

4 The Summary Charts

In the Néron model of an elliptic curve the special fiber is unique and is called the reduction type of the elliptic curve [NÉ]. In this section we take an alternate approach and define a reduction type of a Weierstrass equation by imposing conditions on the a_i. We do this separately for the three cases, depending on if the residue characteristic is 2, 3, or $\neq 2, 3$. We could say that *Type* is a function of the a_i as follows

$$(a_1, a_2, a_3, a_4, a_6) \epsilon R^5 \rightarrow \{Types\}. \tag{11}$$

Since not every Weierstrass equation meets the conditions for one of the reduction types, we must include "None" in the range of the function.

We can define this type function by describing the inverse image of each type. That is, we make requirements on the a_i for each type. These requirements involve the valuations $v(a_i)$, $v(d)$, and the number of distinct roots of certain auxiliary polynomials with coefficients determined by the a_i.

Because the function is mostly determined by the valuations of a_i, and d, we will summarize the types' definitions by using a series of charts depending on the characteristic of the residue field. There will also be sub-charts which define the members of a family of reduction types. For example, such a sub chart tells us which n a reduction type I_n has.

As mentioned, a Weierstrass equation may not meet the conditions for one of the reduction types, and in this case we say that the type is "None". If the 5-tuple a_1, a_2, a_3, a_4, a_6 does have a bona-fide type, the Weierstrass equation will be said to be in chart form. Because the types belonging to a family of reduction types also require a subchart to define them, there is a two step definition for these types. If the 5-tuple a_1, a_2, a_3, a_4, a_6 meets the requirements to be in one of the families of reduction types defined below, but may not necessarily have a type, it will be said to be in pre-chart form.

Later, in section 5 we will see that every Weierstrass equation can be translated into and chart form. We make this precise with a definition.

Definition 4.1 (Chart Form)

Let R be an arbitrary DVR with residue field κ. Let f be a Weierstrass equation defining an elliptic scheme over R.

If $char(\kappa) = 2$ the equation f is In Pre-Chart Form if the valuations $v(a_i)$, $v(b_i)$, $v(d)$, and polynomial conditions meet the requirements of one of the types or families of types described with Chart 4.2.

The equation f is In Chart Form if it is in pre-chart form and either Chart 4.2 does not define a family of reduction types, or if Chart 4.2 does define a family of reduction types and furthermore the valuations $v(a_i)$, $v(b_i)$, $v(d)$, and polynomial conditions meet the requirements of one of the sub-charts Chart 4.3, Chart 4.4 or, Chart 4.5.

If $char(\kappa) = 3$ the equation f is In Pre-Chart Form if the valuations $v(a_i)$, $v(b_i)$, $v(d)$, and polynomial conditions meet the requirements of one of the types or families of types described with Chart 4.9.

The equation f is In Chart Form if it is in pre-chart form and either Chart 4.9 does not define a family of reduction types, or if Chart 4.9 does define a family of reduction types and furthermore the valuations $v(a_i)$, $v(b_i)$, $v(d)$, and polynomial conditions meet the requirements of one of the sub-charts Chart 4.7, or Chart 4.8.

If $char(\kappa) \neq 2, 3$ the equation f is In Pre-Chart Form if the valuations $v(a_i)$, $v(b_i)$, $v(d)$, and polynomial conditions meet the requirements of one of the types or families of types described with Chart 4.6.

The equation f is In Chart Form if it is in pre-chart form and either Chart 4.6 does not define a family of reduction types, or if Chart 4.6 does define a family of reduction types and furthermore the valuations $v(a_i)$, $v(b_i)$, $v(d)$, and polynomial conditions meet the requirements of one of the sub-charts Chart 4.7 or Chart 4.8.

4.1 Chart Notations

Throughout this section the DVR R will have uniformizer π , and residue field κ.

As mentioned above we will define types by imposing conditions on the a_i. The charts provide most of these conditions.

An integer N under the column "Type A" and in the row "$v(a_i)$" means one condition for "Type A" is that $v(a_i) = N$.

A symbol $N+$ under the column "Type A" and in the row "$v(a_i)$" means one condition for "Type A" is that $v(a_i) \geq N$.

A symbol Nns under the column "Type A" and in the row "$v(a_i)$" means one condition for "Type A" is that $v(a_i) = N$, and that $\frac{a_i}{\pi^N}$ is not a square in κ.

A symbol Nnq under the column "Type A" and in the row "$v(a_i)$" means one condition for "Type A" is that $v(a_i) = N$, and that $\frac{a_i}{\pi^N}$ is not a cube in κ.

A blank space in a chart indicates that there is no condition to be met. Following the chart are other conditions that the a_i must satisfy to be a given type. These usually involve the number of roots of a polynomial in $\kappa[X]$ or $\kappa[Y]$.

4.2 Char 2 Residue Field

Suppose the residue characteristic of κ is 2. Here we define the reduction types in terms of the valuations of the a_i, d, and other polynomial conditions on the a_i. Note that the families I_n , I_n^*, or K_n have sub charts. A Weierstrass equation f is in chart form if it meets all of the criteria of one of these types defined here.

For example, f is of type I_0 if $v(d) = 0$.

We define f as one of the types in the I_n family if $v(a_1) = 0$, $v(a_2) \geq 0$, $v(a_3) \geq 1$, $v(a_4) \geq 1$, and $v(a_6) \geq 1$. We consult the subchart Chart 4.3 to determine exactly which type it is.

We define f to be of type $X1$ if $v(a_1) \geq 1$, $v(a_2) \geq 0$, $v(a_3) \geq 1$, $v(a_4) = 0$, $v(a_6) \geq 0$, and a_4 is not a square in κ.

The other types are defined in a similar fashion using the following chart.

Type	I_0	I_n	$X1$	$Y1$	K_n	II	III	IV	$Y2$	I_0^*	$X2$	I_n^*	IV^*	$Y3$	III^*	II^*	o/w
$v(a_1)$	0+	0	1+	1+	1+	1+	1+	1+	1+	1+	1+	1+	1+	1+	1+	1+	1+
$v(a_2)$	0+	0+	0+	0+	$0ns$	1+	1+	1+	1+	1+	1+	1	2+	2+	2+	2+	2+
$v(a_3)$	0+	1+	1+	1+	1+	1+	1+	1	2+	2+	2+	2+	2	3+	3+	3+	3+
$v(a_4)$	0+	1+	$0ns$	1+	1+	1+	1	2+	2+	2+	$2ns$	3+	3+	3+	3	4+	4+
$v(a_6)$	0+	1+	0+	$0ns$	1+	1	2+	2+	$2ns$	3+	3+	4+	4+	$4ns$	5+	5	6+
$v(d)$	0							4					8				12+

(12)

An integer n with 'ns' for a_i means $\frac{a_i}{\pi^n}$ is not a square in κ.

For type I_0^*, the following polynomial in $\kappa[X]$

$$X^3 + \frac{a_2}{\pi}X^2 + \frac{a_4}{\pi^2}X + \frac{a_6}{\pi^3}. \tag{13}$$

must have distinct roots in some extension of κ , where π is a uniformizer in the DVR, and the coefficients of the polynomial are considered as elements of κ.

For type $X2$, the following polynomial in $\kappa[X]$

$$X^3 + \frac{a_2}{\pi}X^2 + \frac{a_4}{\pi^2}X + \frac{a_6}{\pi^3}. \tag{14}$$

must have one rational root, and one double root not defined over κ, where π is a uniformizer in the DVR, and the coefficients of the polynomial are considered as elements of κ. Translating the single rational root to 0, gives us a form with $v(a_2) \geq 2$ and $v(a_6) \geq 4$.

4.3 Char 2 I_n Detail

This is the subchart for the family I_n. When defining some of these types, we reference b_8. This is just the standard polynomial in the a_i.

$$b_8 = a_1^6 a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \tag{15}$$

For example, f to be of type I_2 if $v(a_1) = 0$, $v(a_2) \geq 0$, $v(a_3) \geq 1$, $v(a_4) \geq 1$, $v(a_6) \geq 2$, $v(b_8) = 0$, and $v(d) = 0$.

Type	I_1	I_2	I_3	I_4	I_5	...
$v(a_1)$	0	0	0	0	0	...
$v(a_2)$	0+	0+	0+	0+	0+	...
$v(a_3)$	1+	1+	2+	2+	2+	...
$v(a_4)$	1+	1+	2+	2+	2+	...
$v(a_6)$	1	2+	3	4+	5	...
$v(d)$	1	2	3	4	5	...
$v(b_8)$	1	2	3	4	5	...

(16)

4.4 Char 2 K_n Detail

This is the subchart for the family K_n.

Type	K_1	K_2	K_2'	K_3	K_4	K_4'	K_5	K_6	K_6	...
$v(a_1)$	1+	1+	1+	1+	1+	1+	1+	1+	1+	...
$v(a_2)$	$0ns$	$0ns$	$0ns$	$0ns$	$0ns$	$0ns$	$0ns$	$0ns$	$0ns$	...
$v(a_3)$	1+	1+	2+	2+	2+	2+	3+	3+	4+	...
$v(a_4)$	1+	1+	2+	2+	2+	2+	3+	3+	4+	...
$v(a_6)$	1	2+	$2nt$	3	4+	$4nt$	5	6+	$6nt$	...
$v(b_8)$		2			4			6		...

(17)

For types K_n' we also demand that the quadric

$$y^2 = a_2x^2 + \frac{a_6}{\pi^n} \tag{18}$$

has no rational points in κ . This quadric is then a non reduced double line. The 'nt' stands for non translatable.

4.5 Char 2 I_n^* Detail

This is the subchart for the family I_n^*.

$Type$	I_1^*	$T1$	I_2^*	$T2$	I_3^*	$T3$	I_4^*	$T4$	...
$v(a_1)$	1+	1+	1+	1+	1+	1+	1+	1+	...
$v(a_2)$	1	1	1	1	1	1	1	1	...
$v(a_3)$	2	3+	3+	3+	3	4+	4+	4+	...
$v(a_4)$	3+	3+	3	4+	4+	4+	4	5+	...
$v(a_6)$	4+	$4ns$	5+	$5ns$	6+	$6ns$	7+	$7ns$	...

(19)

For types T_n with n odd we demand that the $\frac{a_6}{\pi^{n+3}}$ is not square in κ.

For types T_n with n even we demand that the $\frac{a_6}{a_2\pi^{n+2}}$ is not square in κ.

4.6 Char $\neq 2, 3$ General Form

Suppose the residue characteristic of κ is not 2 or 3. Here we define the reduction types in terms of the valuations of the a_i, d, and other polynomial conditions. Note the families I_n , I_n^*, or K_n have sub charts. A Weierstrass equation is in form if it meets the criteria of one of these types here.

In this case we will assume that the residue characteristic is not 2 or 3, but we will not divide by 2 to complete the square. This chart does not give us any new types, but it shows just the translations that are demanded by Tate's algorithm .

$Type$	I_0	I_n	II	III	IV	I_0^*	I_n^*	IV^*	III^*	II^*	o/w
$v(a_1)$		0+	1+	1+	1+	1+	1+	1+	1+	1+	1+
$v(a_2)$		0+	1+	1+	1+	1+	1	2+	2+	2+	2+
$v(a_3)$		1+	1+	1+	1+	2+	2+	2+	3+	3+	3+
$v(a_4)$		1+	1+	$1 =$	2+	2+	3+	3+	3	4+	4+
$v(a_6)$		1+	$1 =$	2+	2+	3+	4+	4+	5+	$5 =$	6+
$v(d)$	0	n	2	3	4	6	$6+n$	8	9	10	12+

(20)

For type I_n^*, we must also require $a_1^2 + 4a_2$ to be a unit.

For type IV we require the polynomial

$$X^2 + \frac{a_3}{\pi}X + \frac{a_6}{\pi^2} \tag{21}$$

to have distinct roots in some extension of κ.

For type IV^* we require the polynomial

$$X^2 + \frac{a_3}{\pi^2}X + \frac{a_6}{\pi^4} \tag{22}$$

to have distinct roots in some extension of κ.

For type I_0^*, we must also require the polynomial

$$X^3 + \frac{a_2}{\pi}X^2 + \frac{a_4}{\pi^2}X + \frac{a_6}{\pi^3} \tag{23}$$

to have distinct roots in some extension of κ.

4.7 Char $\neq 2, 3$ I_n General Form Detail

This subchart for the family I_n^* would be the same chart as the Characteristic 2 I_n Detail Chart 4.3 , except we allow $v(a_1) \geq 0$ and demand that $a_1^2 + 4a_2$ is a unit.

This analysis is also valid for all characteristics $\neq 2$, even for non perfect residue fields.

4.8 Char $\neq 2, 3$ I_n^* general form detail

This is the subchart for the family I_n^*. This analysis is also valid for all characteristics $\neq 2$, even for non perfect residue fields.

Type	I_1^*	I_2^*	I_3^*	I_4^*	I_5^*...	
$v(a_1)$	1+	1+	1+	1+	1+	...
$v(a_2)$	1	1	1	1	1	...
$v(a_3)$	2+	3+	3+	4+	4+	...
$v(a_4)$	3+	3+	4+	4+	5+	...
$v(a_6)$	4+	5+	6+	7+	8+	...
$v(d)$	7	8	9	10	11	...

(24)

For type I_n^* with n odd we require the polynomial

$$X^2 + \frac{a_3}{\pi^{\frac{n+3}{2}}} X + \frac{a_6}{\pi^{n+3}} \tag{25}$$

to have distinct roots in some extension of κ

For type I_n^* with n even we require the polynomial

$$\frac{a_2}{\pi} X^2 + \frac{a_4}{\pi^{\frac{n+4}{2}}} X + \frac{a_6}{\pi^{n+3}} \tag{26}$$

to have distinct roots in some extension of κ

4.9 Char 3 Residue Field

In this case we will complete the square and assume $a_1 = a_3 = 0$. The reason we allow ourselves to divide by two here is that if a DVR obtained by localizing a local ring O has residue characteristic 3, all other localizations of O must have residue characteristic 0 or 3.

Type	I_0	I_n	$Z1$	II	III	IV	I_0^*	$Z2$	I_n^*	IV^*	III^*	II^*	o/w
$v(a_2)$		0	1+	1+	1+	1+	1+	2+	1+	2+	2+	2+	2+
$v(a_4)$		1+	1+	1+	1	2+	2+	3+	3+	3+	3	4+	4+
$v(a_6)$		1+	$0nq$	1	2+	2	3+	$3nq$	4+	4	5+	5	6+
$v(d)$	0	n			3		6		$6+n$		9		12+

(27)

An integer n with 'nq' for a_6 means $\frac{a_6}{\pi^n}$ is not a square in κ . For type I_0^*, we must also require the polynomial

$$X^3 + \frac{a_2}{\pi} X^2 + \frac{a_4}{\pi^2} X + \frac{a_6}{\pi^3} \tag{28}$$

to have distinct roots in some extension of κ .

For details on I_n and I_n^*, see charts 4.3, and 4.8 with $a_1 = a_3 = 0$.

This concludes the definition of the reduction types of a Weierstrass equation and the criteria for a Weierstrass equation to be in chart form or pre-chart form.

5 Proof of the Charts

In this section we will prove that any Weierstrass equation over a DVR can be translated to a form specified on one of the charts above.

Proposition 5.1 (Translate to Chart Form)
Let R be an arbitrary DVR with residue field κ. Let f be a Weierstrass equation defining an elliptic scheme over R. Then there exist R translations of x and y such that the translated Weierstrass equation f' is in chart form.

We continue to use the notation R for the DVR, π for the uniformizer, and κ for the residue field. When working with the valuations of the a_i, we will treat them as elements in R, but we will also use the same notation for their images in the residue field.

5.1 Char 2 Residue Field

Given any Weierstrass equation with coefficients in the DVR R, I will show that there exist R translations $x \mapsto x + \alpha$ and $y \mapsto y + \beta x + \gamma$ such that the coefficients a_i satisfy the conditions of one of the types on Chart 4.2. It is useful to refer to that chart while reading this section.

Write κ for the residue field, and π for a uniformizer in the DVR. We use the fact that a hypersurface defined by $f(x, y) = 0$ with a singularity in the special fiber $\pi = 0$ must have $f = \frac{df}{dx} = \frac{df}{dy} = 0 \pmod{\pi}$. For reference, our Weierstrass equation f is given by

$$y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6) = 0 \tag{29}$$

Suppose $v(d) = 0$. Then we have type I_0.

Suppose instead $v(d) > 0$ and $v(a_1) = 0$. Translate via $x = x' - a_3/a_1$ and compute new a_i. Then $v(a_3) > 0$. The singularity at (x, y) on the special fiber satisfies $\frac{df}{dy} = 2y + a_1x + a_3 = 0 \pmod{\pi}$. So x=0 at the singular point.

Now translate via $y = y' + a_4/a_1$, and compute new a_i. Now $v(a_4) > 0$. The singularity at (x, y) on the special fiber satisfies $\frac{df}{dx} = a_1 y + a_4{=}0 \pmod{\pi}$. So y=0 at the singular point. Finally $f = 0 (\bmod\ \pi)$ implies that $v(a_6) > 0$. So in this case $v(a_1) = 0, v(a_3) > 0, v(a_4) > 0, v(a_6) > 0$. We are in one of the cases I_n. We consult a following section 5.2] to determine which type it is.

Now suppose $v(d) > 0$ and $v(a_1) > 0$. Then $\frac{df}{dy} = 0 \pmod{\pi}$ implies $2y + a_1 x + a_3 = a_3 = 0 (\bmod\ \pi)$. So $v(a_3) > 0$.

Assume additionally that a_4 is not a square in κ. Then we are in the case $X1$.

Otherwise assume that a_4 is a square in κ. Now translate via $x = x' + \alpha$ where $\alpha^2 = a_4 (\bmod\ \pi)$ and compute new a_i. Now $v(a_4) > 0$. If a_6 is not a square in the residue field we are in the case $Y1$.

Otherwise assume additionally that a_6 is a square in κ. Now translate via $y = y' + \alpha$ where $\alpha^2 = a_6 \pmod{\pi}$ and compute new a_i. Now $v(a_6) > 0$.

Assume that a_2 is not a square in the residue field. Then we are in one of the cases K_n, or K'_n. We consult a following section 5.3 to determine which type it is.

Otherwise assume additionally that a_2 is a square in κ. Now translate via $y = y' + \beta x$ where $\beta^2 = a_2 \pmod{\pi}$ and compute new a_i. Now $v(a_2) > 0$.

So far, the cumulative translations have given $v(a_1) \geq 1, v(a_2) \geq 1, v(a_3) \geq 1, v(a_4) \geq 1, v(a_6) \geq 1$.

Suppose $v(a_6) = 1$. Then we are in the case II.

Suppose instead $v(a_6) \geq 2$, and $v(a_4) = 1$. Then we are in the case III.

Suppose instead $v(a_4) \geq 2$, and $v(a_3) = 1$. Then we are in the case IV.

Suppose instead $v(a_3) \geq 2$, and $\frac{a_6}{\pi^2}$ is not a square in the residue field. Then we are in the case $Y2$.

Otherwise assume additionally that $\frac{a_6}{\pi^2}$ is a square in κ. Now translate via

$y = y' + \alpha$ where $\alpha^2 = \frac{a_6}{\pi^2} \pmod{\pi}$ and compute new a_i. Now $v(a_6) \geq 3$.

So far, the cumulative translations have given $v(a_1) \geq 1, v(a_2) \geq 1, v(a_3) \geq 2, v(a_4) \geq 3, v(a_6) \geq 3$.

So by reducing $\frac{a_2}{\pi}$, $\frac{a_4}{\pi^2}$, and $\frac{a_6}{\pi^3}$ mod π, we may form the polynomial in $\kappa[X]$:

$$F(X) = X^3 + \frac{a_2}{\pi}X^2 + \frac{a_4}{\pi^2}X + \frac{a_6}{\pi^3}. \tag{30}$$

Assume additionally that $F(X)$ has distinct roots. Then we are in the case I_0^*. Otherwise $F(X)$ has multiple roots. At such a root $F(X) = \frac{dF}{dx} = 0$. Thus one way to check if F has a multiple root is to solve $F(X) = \frac{dF}{dx} = 0$. So there is a multiple root if and only if $v(a_2 a_4 - a_6) > 3$.

Suppose instead that $F(X)$ has a double root and a single root, and that the double root is not rational over κ. Then $F(X)$ factors as $(X - \alpha)(X^2 - \beta)$ with β not a square in κ. This implies $v(a_4) = 2$, and $\beta = \frac{a_4}{\pi^2}$ is not a square in κ. Then we are in the case $X2$.

Optionally, translate the other root to 0, to force $v(a_2) \geq 2$, and $v(a_6) \geq 4$.

Suppose instead that $F(X)$ has a double root and a single root, and that the double root is rational over κ. Then $F(X)$ factors as $(X - \alpha)(X - \beta)^2$. Now translate via $x = x' + \beta\pi$, and compute new a_i. The new $F(X)$ factors as $(X - \alpha')X^2$, with $\alpha' \neq 0$. Now $v(a_4) \geq 3$ and $v(a_6) \geq 4$, and $v(a_2) = 1$. We are in one of the cases I_n^*, or T_n. We consult a following section 5.4 to determine which type it is.

Suppose instead that $F(X)$ has a triple root. A triple root of a cubic in characteristic 2 must always be rational, so $F(X)$ factors as $(X - \alpha)^3$ with $\alpha \epsilon \kappa$. Now translate via $x = x' + \alpha\pi$, and compute new a_i. The new $F(X)$ factors as X^3, so $v(a_4) \geq 3$ and $v(a_6) \geq 4$, and $v(a_2) \geq 2$.

Suppose additionally $v(a_3) = 2$. Then we are in the case IV^*. Otherwise $v(a_3) \geq 3$

Suppose instead $v(a_3) \geq 3$, and $\frac{a_6}{\pi^4}$ is not a square in the residue field. Then we are in the case $Y2$.

Suppose instead $\frac{a_6}{\pi^4}$ is a square in κ. Now translate via $y = y' + \alpha$ where $\alpha^2 = \frac{a_6}{\pi^4} \pmod{\pi}$ and compute new a_i. Now $v(a_6) \geq 5$.

So far, the cumulative translations have given $v(a_1) \geq 1, v(a_2) \geq 2, v(a_3) \geq 3, v(a_4) \geq 3, v(a_6) \geq 5$.

Suppose $v(a_4) = 3$. Then we are in the case III^*. Otherwise $v(a_4) \geq 4$.

Suppose instead $v(a_4) \geq 4$, and $v(a_6) = 5$. Then we are in the case II^*. Otherwise $v(a_6) \geq 6$.

Suppose instead $v(a_6) \geq 6$. Then we have $v(a_1) \geq 1, v(a_2) \geq 2, v(a_3) \geq 3, v(a_4) \geq 4$, and $v(a_6) \geq 6$. We then replace the Weierstrass equation with a more minimal one via the transformation $a_i' = \frac{a_i}{\pi^i}$.

The Weierstrass equation has been transformed to one of the forms on the chart, and we are done unless we had a type in one of the cases I_n, K_n, K_n', I_n^*, or T_n. In any of these cases we consult one of the following sections to determine exactly which type we have. There may be more translations in these subcases.

5.2 Char 2 I_n Detail

This section continues the analysis of the previous section 5.1 in the cases where we have a Weierstrass equation with

$$v(a_1) = 0, v(a_3) > 0, v(a_4) > 0, v(a_6) > 0. \tag{31}$$

We show here that in all of these cases we have a type I_n for some n. In other words, there exist R translations of x and y such that the coefficients a_i satisfy the conditions of one of the types on Chart 4.3.

Suppose $v(a_6) = 1$. Then we have type I_1.

Suppose instead $v(a_6) \geq 2$. Consider the quadratic

$$y^2 + a_1 xy + \frac{a_3}{\pi} y = a_2 x^2 + \frac{a_4}{\pi} x + \frac{a_6}{\pi^2} \tag{32}$$

Let b_8 be defined as in 15. A routine check using the quadratic formula shows that the quadric is degenerate if and only if $v(b_8) > 2$.

Suppose $v(b_8) = 2$ then we are in case I_2.

Suppose instead $v(b_8) > 2$. Then the quadratic is degenerate. Since $b_2 = a_1^2 + 4a_2$ is a unit, the quadratic consist of two distinct intersecting lines.

The singular point where the two lines intersect is a rational point. We see this by computing $\frac{df}{dx} \pmod{\pi}$ and $\frac{df}{dy} \pmod{\pi}$, we see that modulo π it must be at

$$a_1 y = \frac{a_4}{\pi} \tag{33}$$

$$a_1 x = -\frac{a_3}{\pi} \tag{34}$$

Since a_1 is a unit, we can solve for x and y.

Since the singular point is rational, translate it to (0,0) via the translations $y = y' - \frac{a_4}{\pi a_1}$ and $x = x' + \frac{a_3}{\pi a_1}$, and compute new a_i

Now $v(a_3) > 1, v(a_4) > 1, v(a_2) > 2$.

Suppose $v(a_6) = 3$. Then we are in case I_3.

Suppose instead $v(a_6) \geq 4$. Then we form an analogous quadric and check to see if it is degenerate or not. In this process we will be looking successively at the valuations a_6, b_8, and translating the singularities of the general quadratic

$$y^2 + a_1 xy + \frac{a_3}{\pi^{\frac{n}{2}}} y = a_2 x^2 + \frac{a_4}{\pi^{\frac{n}{2}}} x + \frac{a_6}{\pi^n} \tag{35}$$

to (0,0).

So all Weierstrass equations with $v(a_1) = 0, v(a_3) > 0, v(a_4) > 0, v(a_6) > 0$ can be translated to be one of the I_n types on the chart.

5.3 Char 2 K_n Detail

This section continues the analysis of the previous section 5.1 in the cases where we have a Weierstrass equation with $v(a_1) > 0$, $v(a_2) = 0$, $v(a_3) \geq 1, v(a_4) \geq 1, v(a_6) \geq 1$, and a_2 is not a square (mod π). We show here that in all of these cases we have a type K_n for some n, or a type K_n' for some even n. In other words, there exist R translations of x and y such that the coefficients a_i satisfy the conditions of one of the types on Chart 4.4.

Suppose $v(a_6) = 1$. Then we have a type $K1$.

Suppose instead $v(a_6) > 1$.

Consider the quadratic

$$y^2 + a_1xy + \frac{a_3}{\pi}y = a_2x^2 + \frac{a_4}{\pi}x + \frac{a_6}{\pi^2} \tag{36}$$

As above we use b_8 as defined as in 15. Again a routine check using the quadratic formula shows that the quadric is degenerate if and only if $v(b_8) > 2$. We remark that $b_8 = a_2a_3^2 - a_4^2 \bmod \pi^3$.

Suppose further that $v(b_8) = 2$. Then we have a type $K2$.

Suppose instead that $v(b_8) > 2$. Then $a_2a_3^2 = a_4^2 \bmod \pi^3$. Since a_2 in not a square (mod π), we have $v(a_3) \geq 2$ and $v(a_4) \geq 2$. Then the quadratic is

$$y^2 = a_2x^2 + \frac{a_6}{\pi^2} \quad (mod\ \kappa). \tag{37}$$

Suppose further that this quadratic has no κ rational points. Then we have type K_2'.

Suppose instead that (x_0, y_0) is a κ rational point. Then translate this point it to (0,0) via the translations $y = y' - y_0'$ and $x = x' - x_0'$ where x_0' and y_0' are liftings of x_0 and y_0 to the DVR R. Now compute new a_i. We see in particular that $v(a_6) \geq 3$.

Suppose now $v(a_6) = 3$. Then we have a type $K3$.

Suppose instead $v(a_6) > 3$. Then we form an analogous quadric and check to see if it is degenerate or not. In this process we will be looking successively at the valuations a_6, b_8, and checking whether or not

$$y^2 = a_2 x^2 + \frac{a_6}{\pi^n} \tag{38}$$

has a rational point.

So we conclude that all Weierstrass equations with $v(a_1) > 0$, $v(a_2) = 0$, $v(a_3) \geq 1, v(a_4) \geq 1, v(a_6) \geq 1$, and a_2 is not a square (mod π), can be translated to be one of the K_n or K'_n types on the chart.

5.4 Char 2 I^*n Detail

This section continues the analysis of the previous section 5.1 in the cases where we have a Weierstrass equation with $v(a_1) > 0$, $v(a_2) = 1$, $v(a_3) \geq 2, v(a_4) \geq 3, v(a_6) \geq 4$. We show here that in all of these cases we have a type I_n^* for some n, or a type T_n for some n. In other words, there exist R translations of x and y such that the coefficients a_i satisfy the conditions of one of the types on Chart 4.5.

Suppose $v(a_3) = 2$. Then we have a type I_1^*.

Suppose instead that $v(a_3) \geq 3$. Suppose additionally that $\frac{a_6}{\pi^4}$ is not square (mod π). Then we have type $T1$.

Suppose instead that $\frac{a_6}{\pi^4} = \alpha^2$ (mod π). Then translate via $y = y' + \pi^2\alpha$, and compute new a_i. Then $v(a_6) \geq 5$.

Suppose now that $v(a_4) = 3$. Then we have a type I_2^*.

Suppose instead that $v(a_4) \geq 4$. Suppose additionally that $\frac{a_6}{a_2\pi^4}$ is not square (mod π). Then we have type $T2$.

Suppose instead that $\frac{a_6}{a_2\pi^4} = \alpha^2$ (mod π). Then translate via $x = x' - \pi^2\alpha$, and compute new a_i. Then $v(a_6) \geq 6$.

Suppose now that $v(a_3) = 3$. Then we have a type I_3^*.

Suppose instead that $v(a_3) \geq 4$. We continue this process by looking successively at a_3, a_6, a_4, and $\frac{a_6}{a_2}$.

So we conclude that all Weierstrass equations with $v(a_1) > 0$, $v(a_2) = 1$, $v(a_3) \geq 2, v(a_4) \geq 3, v(a_6) \geq 4$ can be translated to be one of the I_n^* or T_i types on the chart.

5.5 Char $\neq 2,3$ General Form

Given any Weierstrass equation with coefficients in the DVR R, I will show that there exist R translations $x \mapsto x + \alpha$ and $y \mapsto y + \beta x + \gamma$ such that the coefficients a_i satisfy the conditions of one of the types on Chart 4.6.

I remark that the content of this section is contained in Tate's original exposition, except that he did not make a chart or the valuations of the a_i. Remember that since $char(\kappa) \neq 2,3$, we could reduce our Weierstrass equation to the simpler form

$$y^2 = x^3 + a_4 x + a_6. \tag{39}$$

But we do not do this. A chart relating the valuations of the a_i to the reduction type for this simpler Weierstrass equation is available as an exercise in [SIL 1]. Although the proof in this section is similar to the characteristic 2 case above, we spell out the details.

Write κ for the residue field, and π for a uniformizer in the DVR. We use the fact that a hypersurface defined by $f(x,y) = 0$ with a singularity in the special fiber $\pi = 0$ must have $f = \frac{df}{dx} = \frac{df}{dy} = 0 \pmod{\pi}$. For reference, our Weierstrass equation f is given by

$$y^2 + a_1 xy + a_3 y - (x^3 + a_2 x^2 + a_4 x + a_6) = 0 \tag{40}$$

Suppose $v(d) = 0$. Then we have type I_0.

Suppose instead $v(d) > 0$. So the Weierstrass equation defines a singular curve over κ.

In order to show that the singular point is κ rational, temporarily complete the square and consider the simpler equation $y'^2 = f(x) \bmod \pi$. The singularity on this curve over κ must have coordinates $(x_0, 0)$ where x_0 is a multiple root of $f(x)$. Since κ does not have characteristic 2 or 3, the multiple root of the cubic polynomial $f(x)$ must lie in κ. Since completing the square was just a κ rational translation of y, we see that the singular point must be given by $x = \alpha$ and $y = \beta$ for $\alpha, \beta \epsilon \kappa$.

Lift α and β to the DVR R, and make a translation via $x = x' + \alpha$ and $y = y' + \beta$, and compute new a_i. Now $v(a_3) \geq 1$,$v(a_4) \geq 1$, and $v(a_6) \geq 1$.

Suppose further that $b_2 = a_1^2 + 4a_2$ is a unit. We are in one of the cases I_n. We consult a following section 5.6 to determine which type it is.

Suppose instead that $v(b_2) > 0$. This means that the polynomial

$$Y^2 + a_1 XY - a_2 X^2 \tag{41}$$

has a double root mod π. Since $char(\kappa) \neq 2$, the root must be in κ. Let α be a lifting of the root to R. Now translate via $y' = y + \alpha x$, and compute new a_i. Now $v(a_1) \geq 1$, and $v(a_2) \geq 1$.

Suppose further that $v(a_6) = 1$. Then we are in the case II.

Suppose instead $v(a_6) \geq 2$, and $v(a_4) = 1$. Then we are in the case III.

Suppose instead $v(a_4) \geq 2$, and that the polynomial

$$Y^2 + \frac{a_3}{\pi} Y - \frac{a_6}{\pi^2} \tag{42}$$

has distinct roots mod π. Then we are in the case IV.

Otherwise suppose the polynomial has a double root mod π. Since $char(\kappa) \neq 2$, the root must be in κ. Let α be a lifting of the root to R. Now translate via $y' = y - \alpha\pi$, and compute new a_i. Now $v(a_3) \geq 2$, and $v(a_6) \geq 3$.

Suppose further that the polynomial

$$X^3 + \frac{a_2}{\pi} X^2 + \frac{a_4}{\pi^2} X + \frac{a_6}{\pi^3} \tag{43}$$

has distinct roots mod π. Then we are in the case I_0^*.

Suppose instead that the polynomial has a double or triple root mod π. Since $char(\kappa) \neq 2, 3$, the root must be in κ. Let α be a lifting of the root to R.

Now translate via $x' = x - \alpha\pi$, and compute new a_i. Now $v(a_4) \geq 3$, and $v(a_6) \geq 4$, since at least two of the three roots have been translated to 0.

Suppose further that $v(a_2) = 1$, or equivalently that the above polynomial had only a double root. We are in one of the cases I_n^*. We consult a following section 5.7 to determine which type it is.

Suppose instead that $v(a_2) \geq 2$, and additionally that the polynomial

$$Y^2 + \frac{a_3}{\pi^2}Y - \frac{a_6}{\pi^4} \tag{44}$$

has distinct roots mod π. Then we are in the case IV^*. Otherwise suppose the polynomial has a double root mod π. Since $char(\kappa) \neq 2$, the root must be in κ. Let α be a lifting of the root to R. Now translate via $y' = y - \alpha\pi^2$, and compute new a_i. Now $v(a_3) \geq 3$, and $v(a_6) \geq 5$.

Suppose further that $v(a_4) = 3$. Then we are in the case III^*.

Suppose instead $v(a_4) \geq 4$, and $v(a_6) = 5$. Then we are in the case II^*.

Suppose instead $v(a_6) \geq 6$. Then we have $v(a_1) \geq 1, v(a_2) \geq 2, v(a_3) \geq 3, v(a_4) \geq 4$, and $v(a_6) \geq 6$. We then replace the Weierstrass equation with a more minimal one via the transformation $a_i' = \frac{a_i}{\pi^i}$.

The Weierstrass equation has been transformed to one of the forms on the chart, and we are done unless we had a type in one of the cases I_n or I_n^*. In these cases we consult one of the following sections to determine exactly which type we have. There may be more translations in these subcases.

I remark again that this is just a restatement of Tate's original algorithm, and we therefore obtain no new fiber types.

5.6 Char $\neq 2, 3$ I_n General Form Detail

This section continues the analysis of the previous section 5.5 in the cases where we have a Weierstrass equation with

$$v(a_1) \geq 0, v(a_2) \geq 0, v(a_3) > 0, v(a_4) > 0, v(a_6) > 0,$$

and $b_2 = a_1^2 + 4a_2$ is a unit.

We show here that in all of these cases we have a type I_n for some n. In other words, there exist R translations of x and y such that the coefficients a_i satisfy the conditions of one of the types on Chart 4.7.

Suppose $v(a_6) = 1$. Then we have type I_1.

Suppose instead $v(a_6) \geq 2$. Consider the quadratic

$$y^2 + a_1xy + \frac{a_3}{\pi}y = a_2x^2 + \frac{a_4}{\pi}x + \frac{a_6}{\pi^2} \tag{45}$$

Let b_8 be defined as in 15. A routine check using the quadratic formula shows that the quadric is degenerate if and only if $v(b_8) > 2$.

Suppose $v(b_8) = 2$ then we are in case I_2, and we can also check that $v(d) = 2$.

Suppose instead $v(b_8) > 2$. Then the quadratic is degenerate. Since $b_2 = a_1^2 + 4a_2$ is a unit, the quadratic consist of two distinct intersecting lines.

The singular point where the two lines intersect is a rational point. We see this by computing $\frac{df}{dx} \pmod{\pi}$ and $\frac{df}{dy} \pmod{\pi}$, we see it must be at $a_1y = 2a_2x + \frac{a_4}{\pi} \pmod{\pi}$, and $2y + a_1x = -\frac{a_3}{\pi} \pmod{\pi}$.

Given the fact that $a_1^2 + 4a_2$ is a unit, it is routine to conclude that the solution of the above two equations is $x = \alpha$, $y = \beta$ for $\alpha, \beta \epsilon \kappa$. In fact we have

$$b_2x = -\frac{a_1a_3 + 2a_4}{\pi} \tag{46}$$

$$b_2y = \frac{a_1a_4 - 2a_2a_3}{\pi} \tag{47}$$

Now since the singular point is rational, translate it to (0,0) via the translations $y = y' + \alpha$ and $x = x' + \beta$, and compute new a_i

Now $v(a_3) > 1, v(a_4) > 1, v(a_2) > 2$. Suppose additionally that $v(a_6) = 3$. Then we are in case I_3.

Suppose instead $v(a_6) \geq 4$. Then we form an analogous quadric and check to see if it is degenerate or not. In this process we will be looking successively at the valuations a_6, b_8, and translating the singularities of the general quadratic

$$y^2 + a_1xy + \frac{a_3}{\pi^{\frac{n}{2}}}y = a_2x^2 + \frac{a_4}{\pi^{\frac{n}{2}}}x + \frac{a_6}{\pi^n} \quad (48)$$

to (0,0).

So all Weierstrass equations with $v(d) > 0, v(a_1) \geq 0, v(a_2) \geq 0, v(a_3) > 0, v(a_4) > 0, v(a_6) > 0$, and $v(b_2) = 0$ can be translated to be one of the I_n types on the chart.

5.7 Char $\neq 2, 3$ I_n^* General Form Detail

This section continues the analysis of the previous section 5.5 in the cases where we have a Weierstrass equation with $v(a_1) > 0$, $v(a_2) = 1$, $v(a_3) \geq 2, v(a_4) \geq 3, v(a_6) \geq 4$. We show here that in all of these cases we have a type I_n^* for some n. In other words, there exist R translations of x and y such that the coefficients a_i satisfy the conditions of one of the types on Chart 4.8.

Suppose the quadratic

$$Y^2 + \frac{a_3}{\pi^2}Y + \frac{a_6}{\pi^4} \quad (49)$$

has distinct roots modulo π. In this case we can compute $v(d) = 7$, and so we have a type I_1^*.

Suppose instead that the polynomial has a double root. Since κ does not have characteristic 2 , the double root of the polynomial must lie in κ. Let α be a lifting of the double root to the DVR R. Then translate via $y = y' + \pi^2\alpha$, and compute new a_i.

Then $v(a_3) \geq 3$, and $v(a_6) \geq 5$.

Now suppose now that the quadratic

$$X^2 + \frac{a_4}{\pi^2 a_2}X + \frac{a_6}{\pi^4 a_2} \quad (50)$$

has distinct roots modulo π. In this case we can compute $v(d) = 8$, and so we have a type I_2^*.

Suppose instead that the polynomial has a double root. Since κ does not have characteristic 2 , the double root of the polynomial must lie in κ. Let α be a lifting of the double root to the DVR R. Then translate via $x = x' + \pi^2\alpha$, and compute new a_i.

Then $v(a_4) \geq 4$, and $v(a_6) \geq 6$.

We continue the process by considering for higher integers k the polynomials in κ of the form

$$Y^2 + \frac{a_3}{\pi^k}Y + \frac{a_6}{\pi^{2k}} \tag{51}$$

and the polynomials in κ of the form

$$X^2 + \frac{a_4}{\pi^k a_2}X + \frac{a_6}{\pi^{2k}a_2}. \tag{52}$$

Since $v(d)$ is finite, the process eventually terminates. So we conclude that all Weierstrass equations with $v(a_1) > 0$, $v(a_2) = 1$, $v(a_3) \geq 2, v(a_4) \geq 3, v(a_6) \geq 4$ can be translated to be one of the I_n^* types on the chart. In fact we obtain a type I_n^* exactly when $v(d) = n + 3$.

5.8 Char 3 Residue Field

In this case where κ has characteristic 3, we are going to complete the square and assume that our Weierstrass equation is of the form

$$y^2 = f(x) = x^3 + a_2x^2 + a_4x + a_6. \tag{53}$$

Given any Weierstrass equation of this form with coefficients in the DVR R, I will show that there exist R translations $x \mapsto x + \alpha$ such that the coefficients a_i satisfy the conditions of one of the types on Chart 4.9.

Suppose $v(d) = 0$. Then we have type I_0.

Suppose instead $v(d) > 0$. Then $f(x)$ has a multiple root.

Suppose $f(x)$ has a triple root α which is not rational over κ. Then α lies in a degree three non separable extension of κ. So

$$f(x) = (x - \alpha)^3 = x^3 + 3\alpha x^2 + 3\alpha^2 x + \alpha^3 \tag{54}$$

This means $v(a_2) \geq 1$, $v(a_4) \geq 1$, and a_6 is a unit but not a cube in the residue field. In this case we are in the case $Z1$

Suppose instead that $f(x)$ has a double root and a single root. Since the characteristic of κ is 3, the double root α must be rational over κ. Translate via $x = x' + \alpha$. Then $v(a_2) = 0$, $v(a_4) \geq 1$, and $v(a_6) \geq 1$. We are in one of the cases I_n. We consult the same section 5.6 as for characteristic $\kappa \neq 2, 3$ to determine which type it is.

Suppose instead that f has a triple rational root α which is in κ. Translate via $x = x' + \alpha$ so that $v(a_2) \geq 1$, $v(a_4) \geq 1$, and $v(a_6) \geq 1$.

Suppose additionally that $v(a_6) = 1$. Then we have type II.

Suppose instead that $v(a_6) > 1$ and $v(a_4) = 1$. Then we have type III.

Suppose instead that $v(a_4) > 1$ and $v(a_6) = 2$. Then we have type IV.

Suppose instead that $v(a_6) \geq 3$, and examine the polynomial

$$F(X) = X^3 + \frac{a_2}{\pi}X^2 + \frac{a_4}{\pi^2}X + \frac{a_6}{\pi^3}. \tag{55}$$

Suppose $F(X)$ has distinct roots. Then we have type I_0^*. We can also compute $v(d) = 6$ in this case.

Suppose instead that $F(x)$ has a triple root α which is not rational over κ. Then α lies in a degree three non separable extension of κ. So

$$F(X) = (X - \alpha)^3 = X^3 + 3\alpha X^2 + 3\alpha^2 X + \alpha^3 \tag{56}$$

This means $v(a_2) \geq 2$, $v(a_4) \geq 3$, and $v(a_6) = 3$ but $\frac{a_6}{\pi^3}$ is not a cube in the residue field. In this case we are in the case $Z2$.

Suppose instead that $F(X)$ has a double root and a single root. Since the characteristic of κ is 3, the double root α must be rational over κ. Translate via $x = x' + \alpha\pi$. Then $v(a_2) = 1$, $v(a_4) \geq 3$, and $v(a_6) \geq 4$. We are in one of the cases I_n^*. We consult the same section 5.7 as for characteristic $\kappa \neq 2, 3$ to determine which type it is. In that section we see that we obtain a type I_n^* with $n = v(d) - 3$.

Suppose instead that F has a triple rational root α which is in κ. Translate via $x = x' + \alpha\pi$ so that $v(a_2) \geq 2$, $v(a_4) \geq 3$, and $v(a_6) \geq 4$.

Suppose additionally that $v(a_6) = 4$. Then we have type II.

Suppose instead that $v(a_6) > 4$ and $v(a_4) = 3$. Then we have type III.

Suppose instead that $v(a_4) > 3$ and $v(a_6) = 4$. Then we have type IV.

Otherwise $v(a_2) \geq 2, v(a_4) \geq 4$, and $v(a_6) \geq 6$. Then replace the Weierstrass equation with a more minimal one via $a_i' = \frac{a_i}{\pi^i}$.

The Weierstrass equation has been transformed to one of the forms on the chart, and we are done unless we had a type in one of the cases I_n or I_n^*. In these cases we consult either Section 5.6 or Section 5.7 to determine exactly which type we have and to show that the Weierstrass equation can be translated to be in chart form.

6 New Special Fibers

In this section we use the charts of the previous section to find the regular model of the elliptic scheme over the DVR by performing a series of blow ups. We will prove the following

Theorem 6.1 (Existence of Regular Model)
Let E be an elliptic curve defined by a minimal Weierstrass equation f over a DVR R with not necessarily perfect residue field κ. Then there exist a flat regular projective minimal model $C \to R$ with generic fiber E.

6.1 Kodaira Types

Suppose the valuations of the a_i satisfy the conditions on one of the charts for one of the standard Kodaira types. Then performing exactly the blowups suggested in Tate's algorithm will produce the standard special fiber on Kodaira's list. These types are $I_n, II, III, IV, I_0^*, I_n^*, IV^*, III^*, II^*$. I will not reproduce the blow ups here.

6.2 Blow ups

Many of the blow ups are the same as those prescribed by Tate's algorithm. So be brief, I will specify the ideal such as (x, y, π) that I am blowing up at, but will not compute all coordinate patches. I will show the coordinate patch or patches only where new components in the special fiber emerge, or some order of tangency is shown to exist. I will also start with the affine subscheme of $R[x, y]$ defined by the Weierstrass equation, instead of using the projective ring $R[X, Y, Z]$, since there are no singularities at $Z = 0$. See section 9.2 for some further discussion of blow ups in general.

6.3 Checking Regularity

We are trying to resolve the singularities in the total space X, so after each blow up we will need to check if there are any singularities. First notice that since we are starting with a generically non singular elliptic curve, any singularities must lie in the special fiber $\pi = 0$. Also, any singular point of the scheme X must also be a singular point in the fiber $\pi = 0$. Of course a singular point in the fiber need not be a singular point in X; for example the cusp in the special fiber of the Kodaira type II is a regular point of the whole space. To check regularity in the total space we compute the dimension of the vector space $\frac{m}{m^2}$

To check regularity in affine space over a field κ, one can pass to an algebraic closure of κ, and use the rank of the Jacobian matrix to determine singularities. In the case of a hypersurface $f = 0$ in $\kappa[x, y]$, this amounts to finding points where

$$f = \frac{df}{dx} = \frac{df}{dy} = 0 \tag{57}$$

Given a singular ideal in the ring obtained by passing to the algebraic closure, we intersect it with the original ring, and check if it is singular.

For example $\kappa[x, y]/(y^2 + x^3 + t)$ with κ a field of characteristic 2 and t not a square in κ has a singularity in the algebraic closure of κ, namely $x = 0, y = \sqrt{(t)}$. The intersection of that ideal with the ring $\kappa[x, y]/(y^2 + x^3 + t)$, is the maximal ideal $(x, y^2 + t)$. But $\frac{m}{m^2}$ is only one dimensional since $y^2 + x^3 + t = y^2 + t$ modulo m^2.

We also remark that fields and the polynomials $\kappa[x]$ over a field κ are local rings, and tensor products of regular rings are again regular. Thus, for example,

$$\kappa[x, y]/(x^2 + t) = \kappa[x]/(x^2 + t) \otimes \kappa[y] \tag{58}$$

is regular when κ a field of characteristic 2 and t not a square in κ.

6.4 Char 3 Z1

We have the subscheme of $R[x, y]$ defined by

$$y^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{59}$$

with $v(a_2) > 0$, $v(a_4) > 0$, $v(a_6) = 0$, a_6 is not a cube in κ. Computing $\frac{d}{dy}$ in the special fiber, we find that the singular locus is must be contained in $\pi = 0$, $2y = 0$. This it must be supported in the subscheme defined by the ideal $(\pi, y, x^3 + a_6)$. Since a_6 is not a cube in κ, the ideal is maximal. Since the polynomial 59 is not zero mod m^2, the vector space $\frac{m}{m^2}$ is two dimensional and the scheme is already regular.

Geometrically, we pass to the algebraic closure of κ, and there the special fiber is a cuspidal cubic with the cusp not rational over κ. The cusp lies in a degree 3 extension of κ.

6.5 Char 3 Z2

We have the subscheme of $R[x, y]$ defined by

$$y^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{60}$$

with $v(a_2) > 1$, $v(a_4) > 2$, $v(a_6) = 3$, $\frac{a_6}{\pi^3}$ is not a cube in κ. The special fiber consists of a multiplicity 1 rational curve.

Blow up at the singular point (x, y, π). For the third coordinate patch put $x = x_1\pi$ and $y = y_1\pi$. This patch is the affine subscheme of $R[x_1, y_1]$ defined by

$$y_1^2 = x_1^3\pi + a_2x_1^2 + \frac{a_4}{\pi}x_1 + \frac{a_6}{\pi^2} \tag{61}$$

The special fiber consists of a multiplicity 2 rational curve defined by $y_1 = 0$ This scheme has singularities, all of which are supported in (y_1, π)

Blow up at the double line (y_1, π). For the second coordinate patch put $y_1 = y_2\pi$. This patch is the affine subscheme of $R[x_1, y_2]$ defined by

$$y_2^2\pi = x_1^3 + \frac{a_2}{\pi}x_1^2 + \frac{a_4}{\pi^2}x_1 + \frac{a_6}{\pi^3} \tag{62}$$

The special fiber consists of $(\pi, x_1^3 + \frac{a_6}{\pi^3})$. This is a multiplicity 3 rational curve in the algebraic closure of κ

The special fiber contains no singular points since $\kappa[x]/(x_1^3 + \frac{a_6}{\pi^3})$ is a field, and the special fiber is that field tensored with $\kappa[y]$. Thus the total space must also be regular.

Geometrically, we pass to the algebraic closure of κ, and there complete special fiber consists of a chain of rational curves of multiplicities 1,2, and 3. One can check that these curves intersect normally. The points on the last component are not rational over κ.

6.6 Char 2 X1

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{63}$$

with $v(a_1) > 0, v(a_3) > 0, v(a_4) = 0$, a_4 is not a square in κ. Computing $\frac{d}{dx}$ in the special fiber, we find that the singular locus is must be contained in $\pi = 0$, $x^2 + a_4$. Combining this with polynomial 63 we also see $y^2 = a_6 - a_2a_4$. Thus any singularity must be supported in the subscheme defined by the ideal $(\pi, x^2 + a_4, y^2 - a_6 - a_2a_4)$. This may or may not be maximal depending on whether $a_6 - a_2a_4$ is a square in κ. If it is a square then let α be the square root. Then the ideal $(\pi, x^3 + a_6, y - \alpha)$ is maximal. In either case let m be the maximal ideal and notice that equation (1) is not zero in $\frac{m}{m^2}$, so the vector space $\frac{m}{m^2}$ is two dimensional and the scheme is already regular.

Geometrically, we pass to the algebraic closure of κ, and there the special fiber is a cuspidal cubic with the cusp not rational over κ. The cusp lies in a degree 2 or 4 extension of κ.

6.7 Char 2 X2

We have the subscheme of $R[x,y]$ defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{64}$$

with $v(a_1) \geq 1, v(a_2) \geq 2, v(a_3) \geq 2, v(a_4) = 2, v(a_6) \geq 4$, $\frac{a_4}{\pi^2}$ is not a square in κ. Here I am assuming the single root of the polynomial $x^3+\frac{a_2}{\pi}x^2+\frac{a_4}{\pi^2}x+\frac{a_6}{\pi^3}$ has been translated to 0. See the note inj section 4.2. The original special fiber consists of a multiplicity 1 rational curve.

Blow up at the singular point (x, y, π). For the third coordinate patch put $x = x_1\pi$ and $y = y_1\pi$. This patch is the affine subscheme of $R[x_1, y_1]$ defined by

$$y_1^2 + a_1x_1y_1 + \frac{a_3}{\pi}y_1 = x_1^3\pi + a_2x_1^2 + \frac{a_4}{\pi}x_1 + \frac{a_6}{\pi^2} \tag{65}$$

The special fiber consists of a multiplicity 2 rational curve defined by $y_1^2 = 0$ This scheme has singularities, all of which are supported in (y_1, π)

Blow up at the double line (y_1, π). For the second coordinate patch put $y_1 = y_2\pi$. This patch is the affine subscheme of $R[x_1, y_2]$ defined by

$$y_2^2\pi + a_1x_1y_2 + \frac{a_3}{\pi}y_2 = x_1^3 + \frac{a_2}{\pi}x_1^2 + \frac{a_4}{\pi^2}x_1 + \frac{a_6}{\pi^3} \tag{66}$$

The special fiber consists of a $(\pi, x_1^3 + \frac{a_4}{\pi^2}x_1)$ This is a multiplicity 1 rational curve (π, x_1), and a multiplicity 2 curve $(\pi, x_1^2 + \frac{a_4}{\pi^2})$. The multiplicity 1 component has no singularities. The multiplicity 2 component also has no singularities since it is the tensor product of the field $\kappa[x_1]/(x_1^2 + \frac{a_4}{\pi^2})$ with $\kappa[y_2]$. Since the special fiber has no singularities, the total space must also be regular.

Geometrically, we pass to the algebraic closure of κ, and there complete special fiber consists multiplicity 1 component meeting a multiplicity 2 component, which in turn meets one multiplicity 1 and 1 multiplicity 2 component. One can check that these curves intersect normally. The last multiplicity 2 component are not rational over κ, but is defined over a degree two extension of κ.

6.7.1 A Degree 2 Base Extension

Lets take a closer look at the $X2$ type when the characteristic of κ is 2. We remarked above that every point of the last multipilicity two component was a regular point. If we were to make a degree two extention of κ, this would not be the case. Consider the field $\kappa[\alpha]$ where α is a square root of $\frac{a_4}{\pi^2}$. Rewriting equation 66 as

$$\pi(y_2^2 + \frac{a_1}{\pi}x_1y_2 + \frac{a_3}{\pi^2}y_2 - \frac{a_2}{\pi^2}x_1^2 - \frac{a_6}{\pi^4}) = x_1(x_1+\alpha)^2 - x_1^2 2\alpha \tag{67}$$

Focus on the line $\pi = 0, x = -\alpha$. There are one or two singulrities on this line depending on the number of roots of the quadratic in y_2

$$y_2^2 + \frac{a_1}{\pi}\alpha y_2 + \frac{a_3}{\pi^2}y_2 - \frac{a_2}{\pi^2}\alpha^2 - \frac{a_6}{\pi^4} + \frac{2}{\pi}\alpha^3. \tag{68}$$

Note that we use the fact that π divides 2.

In fact, at a maximal ideal defined by the quadratic 68 and $\pi = 0, x = -\alpha$, equation 68 is zero modulo m^2. Thus these are the one or two singularities on this multiplicity 2 component. As we have already remarked, they are not κ rational. But this serves to give a better picture of the geometric special fiber.

6.8 Char 2 Y1

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{69}$$

with $v(a_1) > 0, v(a_3) > 0, v(a_4) > 0, v(a_6) = 0$, a_6 is not a square in κ. This situation is similar to the $X1$ case. Computing $\frac{d}{dx}$ in the special fiber, we find that the singular locus is must be contained in $\pi = 0, x = 0$. Combining this with 69 we also see $y^2 = a_6$ Thus it must be supported in the subscheme defined by the ideal $(\pi, x, y^2 - a_6)$. Since a_6 is not a square in κ, the ideal is maximal. Since the polynomial 69 is not zero mod m^2, the vector space $\frac{m}{m^2}$ is two dimensional and the scheme is already regular.

Geometrically, we pass to the algebraic closure of κ, and there the special fiber is a cuspidal cubic with the cusp not rational over κ. The cusp lies in a degree 2 extension of κ.

6.9 Char 2 Y2

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{70}$$

with $v(a_1) \geq 1, v(a_2) \geq 1, v(a_3) \geq 2, v(a_4) \geq 2, v(a_6) = 2$, $\frac{a_6}{\pi^2}$ is not a square in κ. The original special fiber consists of a multiplicity 1 rational curve.

Blow up at the singular point (x, y, π). For the third coordinate patch put $x = x_1\pi$ and $y = y_1\pi$. This patch is the affine subscheme of $R[x_1, y_1]$ defined by

$$y_1^2 + a_1x_1y_1 + \frac{a_3}{\pi}y_1 = x_1^3\pi + a_2x_1^2 + \frac{a_4}{\pi}x_1 + \frac{a_6}{\pi^2} \tag{71}$$

The special fiber consists of a multiplicity 2 rational curve defined by $y_1^2 = \frac{a_6}{\pi^2}$ This component has no singularities since it is the tensor product of the field $\kappa[y_1]/(y_1^2 = \frac{a_6}{\pi^2})$ with $\kappa[x_1]$. Since the special fiber has no singularities, the total space must also be regular.

Geometrically, we pass to the algebraic closure of κ, and there the complete special fiber consists of a chain of rational curves of multiplicities 1 and 2 The second curve is not rational over κ, but is over a degree 2 extension of κ. One can check that these curves intersect normally.

6.10 Char 2 Y3

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{72}$$

with $v(a_1) \geq 1, v(a_2) \geq 2, v(a_3) \geq 3, v(a_4) \geq 3, v(a_6) = 4$, $\frac{a_6}{\pi^4}$ is not a square in κ. The special fiber consists of a multiplicity 1 rational curve.

Blow up at the singular point (x, y, π). For the third coordinate patch put $x = x_1\pi$ and $y = y_1\pi$. This patch is the affine subscheme of $R[x_1, y_1]$ defined by

$$y_1^2 + a_1x_1y_1 + \frac{a_3}{\pi}y_1 = x_1^3\pi + a_2x_1^2 + \frac{a_4}{\pi}x_1 + \frac{a_6}{\pi^2} \tag{73}$$

The special fiber consists of a multiplicity 2 rational curve defined by $y_1^2 = 0$ This scheme has singularities, all of which are supported in (y_1, π)

Blow up at the double line (y_1, π). For the second coordinate patch put $y_1 = y_2\pi$. This patch is the affine subscheme of $R[x_1, y_2]$ defined by

$$y_2^2\pi + a_1x_1y_2 + \frac{a_3}{\pi}y_2 = x_1^3 + \frac{a_2}{\pi}x_1^2 + \frac{a_4}{\pi^2}x_1 + \frac{a_6}{\pi^3} \tag{74}$$

The special fiber consists of the triple line a (π, x_1^3) The scheme is singular at the ideal $(x_1, y_2^2 - \frac{a_6}{\pi^4}, \pi)$. Note this ideal is maximal since $\frac{a_6}{\pi^4}$ is not a square in κ.

Blow up at this ideal. For the first coordinate patch put $x_1b = y_2^2 - \frac{a_6}{\pi^4}$, and $x_1c = \pi$. Then this patch is the affine subscheme of $R[x_1, y_2]$ defined by

$$bc + \frac{a_1}{\pi}cy_2 + \frac{a_3}{\pi^3}y_2c^2 = x_1 + \frac{a_2}{\pi} + \frac{a_4}{\pi^3}c \tag{75}$$

The special fiber $\pi = 0$ consists of the $c = 0, x_1 = 0, y_2^2 - \frac{a_6}{\pi^4}$ component which has multiplicity 4, and the $x_1 = 0, b + \frac{a_1}{\pi}y_2 + \frac{a_3}{\pi^3}y_2c = \frac{a_4}{\pi^3}, y_2^2 - \frac{a_6}{\pi^4}$ which has multiplicity 2. Thus in this coordinate patch the special fiber consists of a multiplicity 4 component intersecting with a multiplicity 2 component. Neither of these rational curves are defined over κ since $\frac{a_6}{\pi^4}$ is not a square in κ

To see this scheme is non singular set $d = b + \frac{a_1}{\pi}y_2 + \frac{a_3}{\pi^3}y_2c - \frac{a_4}{\pi^3}$ and solve for x_1 to see that the scheme is isomorphic to the subscheme of $R[c, d, y_2]$ given by the equations

$$dc^2 = \pi(1 + \frac{a_2}{\pi^2}c) \tag{76}$$

$$y_2^2 = \frac{a_6}{\pi^4} \tag{77}$$

Consider the subscheme of $R[c, d]$ modulo the first equation. It is non singular since $\frac{d}{d\pi} \neq 0$, and the second is regular generically, and in the special fiber

$\pi = 0$ since the $\kappa[y_2]/(y_2^2 - \frac{a_6}{\pi^4})$ is a field. So the scheme defined by 76, and 77 is the tensor product of two regular schemes and thus is regular.

Geometrically, we pass to the algebraic closure of κ, and there the complete special fiber consists of a chain of rational curves of multiplicities 1, 2, 3, 4, and 2 The last 2 curves are not rational over κ, but are over a degree 2 extension of κ. One can check that these curves intersect normally.

6.11 Char 2 K_n, n odd

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{78}$$

with $v(a_1) \geq 1, v(a_2) = 0, v(a_3) \geq \frac{n+1}{2}, v(a_4) \geq \frac{n+1}{2}, v(a_6) = n$, a_2 is not a square in κ.

Suppose $n = 1$, then the scheme is already regular, and has a cusp at (x, y, π).

Otherwise $n \geq 3$, Computing $\frac{d}{dx} = x^2$ in the special fiber, we find that the only singular point on the scheme is (x, y, π). Blowing up at this ideal, we put $x = x_1\pi$, $y = y_1\pi$ for the third coordinate patch. This patch is the affine subscheme of $R[x_1, y_1]$ defined by

$$y_1^2 + a_1x_1y_1 + \frac{a_3}{\pi}y_1 = x_1^3\pi + a_2x_1^2 + \frac{a_4}{\pi}x_1 + \frac{a_6}{\pi^2} \tag{79}$$

The special fiber $\pi = 0$ is the double line $y_1^2 = a_2x_1^2$ The only rational point on this line is (x_1, y_1, π).

We can check that all other points on this line are nonsingular by localizing away from (y). Putting $z = \frac{x_1}{y_1}$, the open subscheme obtained by discarding the point (x_1, y_1, π) is isomorphic to $\kappa[z, y]/z^2 = a_2$. This is isomorphic to the tensor product of the field $\kappa[z]/z^2 = a_2$, and $\kappa[y]$. So all points except (x_1, y_1, π) are nonsingular points.

Thus the potentially singularity is at $m = (x_1, y_1, \pi)$. Supposing $n = 3$, Equation 79 is not zero modulo m^2, so the scheme is regular.

If $n \geq 5$, perform another blow-up at (x_1, y_1, π). Blowing up at this ideal, we put $x_1 = x_2\pi$, $y_1 = y_2\pi$ for the third coordinate patch. This patch is the affine subscheme of $R[x_2, y_2]$ defined by

$$y_2^2 + a_1x_2y_2 + \frac{a_3}{\pi^2}y_2 = x_2^3\pi^2 + a_2x_2^2 + \frac{a_4}{\pi^2}x_2 + \frac{a_6}{\pi^4} \tag{80}$$

The special fiber $\pi = 0$ is the double line $y_2^2 = a_2x_2^2$.

Thus the addition to the special fiber will be another non rational multiplicity 2 component. Again the only potential singularity is at $m = (x_2, y_2, \pi)$. If $n = 5$, we are done but in general, we will need to perform $\frac{n-1}{2}$ blow ups.

Geometrically, we pass to the algebraic closure of κ, and there the complete special fiber consists of multiplicity 1 rational curve connected to a chain of $\frac{n-1}{2}$ multiplicity 2 components, which are not rational over κ, but are over a degree 2 extension of κ.

One can check that these curves intersect normally, and that we have seen that the points of intersection of two components are κ rational, and the last multiplicity 2 component also contains one other rational point.

6.12 Char 2 K_n, n even

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{81}$$

with $v(a_1) \geq 1, v(a_2) = 0, v(a_3) \geq \frac{n}{2}, v(a_4) \geq \frac{n}{2}, v(a_6) \geq n$, a_2 is not a square in κ, and $v(b_8) = n$

Computing $\frac{d}{dx} = x^2$ in the special fiber, we find that the only singular point on the scheme is (x, y, π). Blowing up at this ideal, we put $x = x_1\pi$, $y = y_1\pi$ for the third coordinate patch. This patch is the affine subscheme of $R[x_1, y_1]$ defined by

$$y_1^2 + a_1x_1y_1 + \frac{a_3}{\pi}y_1 = x_1^3\pi + a_2x_1^2 + \frac{a_4}{\pi}x_1 + \frac{a_6}{\pi^2} \tag{82}$$

The special fiber $\pi = 0$ is the quadric

$$y^2 + \frac{a_3}{\pi}y = a_2x^2 + \frac{a_4}{\pi}x + \frac{a_6}{\pi^2} \tag{83}$$

Supposing $n = 2$, $v(b_8) = 2$ implies that this is a smooth quadric. Thus the addition to the special fiber is a multiplicity 1 component, and the scheme is now regular.

If $n \geq 4$, the quadric is degenerate and the special fiber $\pi = 0$ is the double line $y_1^2 = a_2 x_1^2$, The only rational point on this line is (x_1, y_1, π). As in the K_n for n odd, we check that all other points on this line are nonsingular points. But since $n \geq 4$, (x_1, y_1, π) is a singular point, and so we next blow up at that ideal.

As in the previous case, we need a total of $\frac{n}{2}$ blow ups. Each blow up produces an additional multiplicity 2 component, except for the last blow up which produces a multiplicity 1 component, since $v(b_8) = n$ forces the last quadric to be regular.

One can also check that the components intersect normally, except in the case $n = 2$ In this case the entire special fiber is as the Kodaira type II. To see the tangency, just look in the first coordinate patch of the first blow-up.

Suppose $n \geq 4$. Geometrically, we pass to the algebraic closure of κ, and there the complete special fiber consists of one multiplicity 1 rational curve connected to a chain of $\frac{n-2}{2}$ multiplicity 2 rational curves, connected to a last multiplicity 1 rational curve. The multiplicity two components are not rational over κ, but are over a degree 2 extension of κ

6.13 Char 2 K'_n, n even

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \tag{84}$$

with $v(a_1) \geq 1, v(a_2) = 0, v(a_3) \geq \frac{n+2}{2}, v(a_4) \geq \frac{n+2}{2}, v(a_6) = n$, a_2 is not a square in κ, and $v(b_8) \geq n$. We further assume that the quadratic

$$y^2 = a_2 x^2 + \frac{a_6}{\pi^n} \tag{85}$$

has no rational points. Exactly as in the K_n case we blow up at the only singular point on the scheme (x, y, π). As above we repeat this performing a total of $\frac{n}{2}$ blow ups.

However, in this case, we have $v(a_3) \geq \frac{n+2}{2}, v(a_4) \geq \frac{n+2}{2}$, and $v(b_8) > n$, so the last blow up produces the affine subscheme of $R[x_n, y_n]$ defined by

$$y_n^2 + a_1 x_n y_n + \frac{a_3}{\pi^{\frac{n}{2}}} y_n = x_n^3 \pi^{\frac{n}{2}} + a_2 x_n^2 + \frac{a_4}{\pi^{\frac{n}{2}}} x_n + \frac{a_6}{\pi^n} \tag{86}$$

The special fiber $\pi = 0$ is not a smooth quadric, but rather the degenerate

$$y^2 = a_2 x^2 + \frac{a_6}{\pi^n} \tag{87}$$

which by assumption has no rational points. This last component is a double line.

As in the K_n cases we can check that there are no singular points on this double line, unless we make an extension of the residue field. To see this we still make an extension of the residue field to include α with $\alpha^2 = \frac{a_6}{\pi^n}$. Then translate y by α to obtain the scheme defined by $y'^2 = a_2 x^2$. As explained in the K_n case, this has no singularities away from (x, y'). Intersecting this last ideal with the original ring before base extension, we obtain the ideal $(x, y^2 - \frac{a_6}{\pi^n})$. This ideal is maximal and $\frac{m}{m^2}$ is one dimensional, so we finally conclude that there are no singular points in the special fiber and thus no singular points in the total space.

Geometrically, we pass to the algebraic closure of κ, and there the complete special fiber consists of one multiplicity 1 rational curve connected to a chain of $\frac{n}{2}$ multiplicity 2 rational curves. The multiplicity two components are not rational over κ, but are over a degree 2 extension of κ, the last component may contain a singularity, but only in some extension of κ.

6.14 Char 2 T_n, n odd

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \tag{88}$$

with $v(a_1) \geq 1, v(a_2) = 1, v(a_3) \geq \frac{n+5}{2}, v(a_4) \geq \frac{n+5}{2}, v(a_6) = n+3$, $\frac{a_6}{\pi^{n+3}}$ is not a square in κ. The original special fiber consists of a multiplicity 1 rational curve.

Blow up at the singular point (x, y, π). For the third coordinate patch put $x = x_1\pi$ and $y = y_1\pi$. This patch is the affine subscheme of $R[x_1, y_1]$ defined by

$$y_1^2 + a_1x_1y_1 + \frac{a_3}{\pi}y_1 = x_1^3\pi + a_2x_1^2 + \frac{a_4}{\pi}x_1 + \frac{a_6}{\pi^2} \tag{89}$$

The special fiber consists of a multiplicity 2 rational curve defined by $y_1^2 = 0$ This scheme has singularities, all of which are supported in (y_1, π)

Blow up at the double line (y_1, π). For the second coordinate patch put $y_1 = y_2\pi$. This patch is the affine subscheme of $R[x_1, y_2]$ defined by

$$y_2^2\pi + a_1x_1y_2 + \frac{a_3}{\pi}y_2 = x_1^3 + \frac{a_2}{\pi}x_1^2 + \frac{a_4}{\pi^2}x_1 + \frac{a_6}{\pi^3} \tag{90}$$

The special fiber is $x_1^3 + \frac{a_2}{\pi}x_1^2 = 0$. This consists of the multiplicity 1 rational curve $x + \frac{a_2}{\pi} = 0$, and the double line $x_1^2 = 0$. All singularities supported in (x_1, π)

Blow up at the double line (x_1, π). For the second coordinate patch put $x_1 = x_2\pi$. This patch is the affine subscheme of $R[x_2, y_2]$ defined by

$$y_2^2 + a_1x_2y_2 + \frac{a_3}{\pi^2}y_2 = x_2^3\pi^2 + a_2x_2^2 + \frac{a_4}{\pi^2}x_2 + \frac{a_6}{\pi^4} \tag{91}$$

The special fiber is $y_2^2 = \frac{a_6}{\pi^4}$. Supposing $n = 1$, $\frac{a_6}{\pi^4}$ is not a square in κ and so the special fiber is a double line defined over a degree two extension of κ.

Otherwise $n \geq 3$. Blow up along the double line (y_2, π). For the second coordinate patch put $y_2 = y_3\pi$. The addition to the special fiber will be the double line $x_2^2 = 0$. Then blow up along the double line (x_2, π). After repeating this pair of blow ups $\frac{n+1}{2}$ times we have two multiplicity one rational curves connected to a chain of n multiplicity 2 curves. The final coordinate patch is

$$y'^2 + a_1x'y' + \frac{a_3}{\pi^{\frac{n+3}{2}}}y' = x'^3\pi^{\frac{n+3}{2}} + a_2x'^2 + \frac{a_4}{\pi^{\frac{n+3}{2}}}x' + \frac{a_6}{\pi^{n+3}} \tag{92}$$

The special fiber is $y'^2 = \frac{a_6}{\pi^n+3}$. Since $\frac{a_6}{\pi^n+3}$ is not a square in κ, the special fiber is a double line defined over a degree two extension of κ. Furthermore, there are no singular points in this patch of the special fiber, since $\kappa[y']/(y'^2 - \frac{a_6}{\pi^n+3})$ is a field.

Geometrically, we pass to the algebraic closure of κ, and there the complete special fiber consists of two multiplicity 1 rational curve connected to a chain of $n+1$ multiplicity 2 rational curves, the last of which is not rational over κ, but is over a degree 2 extension of κ.

6.15 Char 2 T_n, n even

We have the subscheme of $R[x, y]$ defined by

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \tag{93}$$

with $v(a_1) \geq 1, v(a_2) = 1, v(a_3) \geq \frac{n+4}{2}, v(a_4) \geq \frac{n+6}{2}, v(a_6) = n+3$, $\frac{a_6}{\pi^{n+3}}$ is not a square in κ. The situation is much like the n- odd case, and we use the same blow ups. In other words blow up at the ideal (x, y, π), then at the double lines $(y_1 \pi)$, and $(x_1 \pi)$. We repeat this last pair of blow ups $\frac{n}{2}$ times and so have two multiplicity one components connected to a chain of n multiplicity 2 curves.

This last patch is the subscheme of $R[x', y']$ defined by

$$y'^2 + a_1 x' y' + \frac{a_3}{\pi^{\frac{n+2}{2}}} y' = x'^3 \pi^{\frac{n+2}{2}} + a_2 x'^2 + \frac{a_4}{\pi^{\frac{n+2}{2}}} x' + \frac{a_6}{\pi^{n+2}} \tag{94}$$

The special fiber is the double line $y'^2 = 0$. This line contains singularities, so we need 1 final blow up.

Blow up at the double line (y', π). For the second coordinate patch put $y' = y'' \pi$. This patch is the affine subscheme of $R[x', y'']$ defined by

$$y''^2 \pi + a_1 x' y'' + \frac{a_3}{\pi^{\frac{n+2}{2}}} y'' = x'^3 \pi^{\frac{n+4}{2}} + \frac{a_2}{\pi} x'^2 + \frac{a_4}{\pi^{\frac{n+4}{2}}} x' + \frac{a_6}{\pi^{n+3}} \tag{95}$$

The special fiber is $\frac{a_2}{\pi} x'^2 + \frac{a_6}{\pi^n+3}$. Since $\frac{a_6}{\pi^n+3}$ is not a square in κ, the special fiber is a double line defined over a degree two extension of κ. Furthermore, there are no singular points in this patch of the special fiber, since $\kappa[x']/(\frac{a_2}{\pi}x'^2 + \frac{a_6}{\pi^n+3})$ is a field.

The final special fiber is of the same form as the T_n for n odd. Geometrically, we pass to the algebraic closure of κ, and there the complete special fiber consists of two multiplicity 1 rational curve connected to a chain of $n + 1$ multiplicity 2 rational curves, the last of which is not rational over κ, but is over a degree 2 extension of κ.

7 Summary of New Reduction Types

7.1 The Extended Tate's Algorithm

At this point we have constructed the regular model described in Theorem 6.1. We summarize the findings in this section.

By nature of the construction of the regular model, we have proven that we can easily find the special fiber of the regular model. Thus we have an extension of Tate's algorithm to DVRs with non perfect residue fields.

Theorem 7.1 (Extension of Tate's Algorithm)
Let E be an elliptic curve defined by a minimal Weierstrass equation f over a DVR R with not necessarily perfect residue field κ. The special fiber of the flat regular projective minimal model $C \to R$ of E specified by theorem 6.1 may be found by writing f in chart form and looking up the reduction type in one of the charts of section 4.

The model C may be constructed by Tate's original blow ups [TA], or by the blow ups specified in section 6

Finally, we have a complete list of the special fibers which may arise.

Corollary 7.2 (Special Fibers)
Let $C \to R$ be a regular projective minimal model of an elliptic curve E defined by a minimal Weierstrass equation f over a DVR R with not necessarily perfect residue field κ. Then the special fiber is either one of the special fibers on Kodaira's list, or is one of the special fibers

$$Z1, Z2, X1, X2, Y1, Y2, Y3, K_n, K'_n, T_n \tag{96}$$

constructed in section 6.

7.2 The Special Fibers

In this section I will describe the special fibers that we have computed. By the geometric special fiber we mean the variety

$$C \otimes_R \overline{\kappa} \tag{97}$$

where $\overline{\kappa}$ is an algebraic closure of κ.

A given point on the special fiber may or may not be κ rational. To say that a whole component of the special fiber is not κ rational means that the reduced subscheme corresponding to that component is an algebraic variety defined over $\overline{\kappa}$ which can not be defined over κ.

If the residue field is of characteristic 3, the special fiber is a type described on Kodaira's list or one of the following new types.

• Type $Z1$ is a nodal cubic. The node of the cubic is not a κ rational point.

• Type $Z2$ is a chain of 1-2-3 multiplicity components. The last component is not rational over κ.

If the residue field is of characteristic 2, the special fiber is a type described on Kodaira's list or one of the following new types.

• Type $X1$ is a nodal cubic The node is not rational over κ.

• Type $Y1$ is a nodal cubic. The node is not rational over κ.

• Type $Y2$ is a chain of 1-2 multiplicity components. The last component is not rational over κ.

• Type $X2$ is a chain of 1-2-1 multiplicity components, with an extra multiplicity 2 component intersecting the first multiplicity 2 component. This last component is not rational over κ.

• Type $Y3$ is a chain of 1-2-3-4-2 multiplicity components. The last two components are not rational over κ.

• Type K_n, (n odd) is a chain of 1-2....2 multiplicity components. There are $\frac{n-1}{2}$ multiplicity 2 components, and these are not rational over κ.

• Type K_n, (n even) is a chain of 1-2....2-1 multiplicity components. There are $\frac{n-2}{2}$ multiplicity 2 components, and these are not rational over κ.

• Type K'_n, (n even) is a chain of 1-2....2-2 multiplicity components. There are $\frac{n}{2}$ multiplicity 2 components, and these are not rational over κ. Furthermore, the last component contains no κ rational points.

• Type T_n, (all n) is a chain of 1-2....2 multiplicity components. There are $n+1$ multiplicity 2 components; only the last one is not rational over κ.

7.3 Relationship with Kodaira Types

This section is not strictly necessary to further sections, but it gives some insight into the geometry of the new reduction types.

The new special fibers arose because the residue field κ was not perfect and had characteristic 2 or 3. In each case a polynomial of degree 2 or 3 had multiple roots over some extension field of κ, that were not κ rational. There is an interesting relationship between these new types and on of the standard Kodaira types. If we were to pretend that these non κ rational multiple roots were indeed distinct, we would obtain a standard Kodaira type, and the configuration of components in the new types is exactly as in the Kodaira type but with some components identified.

As an example of how this might happen, consider a Weierstrass equation with coefficients in $Z(t)[[\pi]]$. There is a natural map from this ring to two different DVRs : $F_2(t)[[\pi]]$, and $Q(t)[[\pi]]$. The first has residue field $F_2(t)$, and the second has residue field $Q(t)$. π is a uniformizer for each. Now consider the Weierstrass equation over each of these DVRs and compute the reduction type. If the reduction type of the Weierstrass equation over $F_2(t)[[\pi]]$ is one of the new types then there is some polynomial in $F_2(t)$ with multiple roots that are not defined over $F_2(t)$. However, the corresponding polynomial over $Q(t)$ will have distinct roots. This can be verified case by case, and we summarize the results in a chart.

$F_2(t)$ $\underline{New\ Type}$	$F_2(t)$ $\underline{Config}$	$Q(t)$ $\underline{Standard}$	$Q(t)$ $\underline{Config}$	$\#Comps$ $\underline{Ident}$
$X1$	1	I_0	E	$-$
$X2$	$1-2-1,2$	I_0^*	$1-2-1,1,1$	2
$Y1$	1	I_0	E	$-$
$Y2$	$1-2$	IV	$1-1,1$	2
$Y3$	$1-2-3-4-2$	$IV*$	$1-2-3-21,21$	$2,2$
$K_n\ n\,odd$	$1-2..2$	I_n	$1-1...1$	$\frac{n-1}{2}$
$K_n\ n\,even$	$1-2...2-1$	I_n	$1-1...1$	$\frac{n-2}{2}$
$K_n'\ n\,even$	$1-2...2$	I_n	$1-1...1$	$\frac{n}{2}$
T_i	$1,1-2-...2$	I_n^*	$1,1-2...2-1,1$	2
$Z1\ (char\,3)$	1	I_0	E	$-$
$Z2\ (char\,3)$	$1-2-3$	I_0^*	$1-2-1,1,1$	3

(98)

Here are some additional notes to clarify the chart. An 'E' for a special fiber means that the special fiber is a smooth elliptic curve. Except for the last 2 entries the residue characteristic of κ is 2. The new types for the last 2 entries actually have residue field $F_3(t)$ The $Y3$ type configuration has the two $2-1$ chains of the type IV^* identified as one $4-2$ chain.

8 Higher Dimensional Bases

8.1 Elliptic Schemes

The goal of this paper is to study elliptic schemes over relatively general base schemes. To this end, we work with a base scheme B that is Noetherian, n dimensional, regular, integral and separated. We are going to study elliptic schemes over B that are defined by Weierstrass equations.

Definition 8.1 (Weierstrass Elliptic Scheme)
Let B be a Noetherian, n dimensional, regular, integral, separated scheme. A Weierstrass Elliptic Scheme over B is a subscheme X of $P^2(B)$ given by Weierstrass equations. That is, for each $p \epsilon B$ there exists an affine open $U = Spec(R)$ containing p such over U that the scheme is equal to $R[X, Y, Z]$ modulo

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3. \qquad (99)$$

where $X, Y,$ and Z are projective coordinates, and $a_i \epsilon R$. Furthermore we assume that over the generic point of B, X is a non-singular elliptic curve.

In our analysis we will begin with a Weierstrass elliptic scheme, and construct a birationally equivalent scheme via some blow-up procedures. In particular, we will blow up the base, pull back the family X to the new base and then desingularize the total space. After these blow-ups, our scheme will not necessarily be a subscheme of two dimensional projective space. For this reason we generalize the above definition.

Definition 8.2 (General Elliptic Scheme)
Let B be a Noetherian, n dimensional, regular, integral, separated scheme. A General Elliptic Scheme over B is a subscheme X of $P^N(B)$ for some integer N such that for some open subset $U \subset B$, the pullback of X to U is a Weierstrass Elliptic Scheme.

The reason for some of the assumptions on B are as follows: We assume that B is a regular integral separated scheme in order to have a theory of divisors on B, and we make the Noetherian hypothesis so that only a finite number of blow ups will be needed when we construct a regular model over the whole base scheme. Before examining elliptic schemes locally we define the notion of the discriminant.

Recall that for any Weierstrass equation as in Equation 2 the discriminant is given by a polynomial in the a_i. The formula may be found in [SIL 1]. Over a DVR, the special fiber of a Weierstrass elliptic scheme is singular if and only if the discriminant has positive valuation. Over more general schemes B, we define the discriminant divisor.

Definition 8.3 (Discriminant Divisor)
Let X be a Weierstrass elliptic scheme over B. For each affine open $U_i = Spec(R_i)$ in B consider a Weierstrass equation defining X. Let $d_i \epsilon R_i$ be defined as the discriminant of the Weierstrass equation. The divisor in B defined locally by the (d_i) is called the Discriminant Divisor.

Note that although the Weierstrass equations may not be unique, the discriminant is well defined up to a unit. Therefore the discriminant divisor is well defined. We are going to assume that the discriminant divisor has normal crossings. First we focus on the local picture.

8.2 Local Elliptic Schemes

Recall that we are supposing that the base is regular of dimension n. By this we mean that for each closed point $p \epsilon B$, the local ring O_p is of dimension n. This also means that for each local ring O_p there exist uniformizing parameters. Explicitly:

Definition 8.4 (Uniformizing Parameters)
Let O be a regular local ring of dimension n. The $t_1, ...t_n$ are called uniformizing parameters for O if the t_i that generate $\frac{m}{m^2}$ as a vector space over the residue field at p.

We now make the important assumption

Assumption 8.5 (Discriminant) *Let X be an elliptic scheme over B. We always assume that the discriminant divisor has normal crossings.*

Intuitively, normal crossings means that the components of the divisor intersect transversely. More precisely we define:

Definition 8.6 (Normal Crossings)
Let D be a divisor on a Noetherian, regular, integral, separated scheme B. D has Normal Crossings if for each $p \epsilon B$, D is the divisor (f) of an element $f \epsilon O_p$ of the form

$$f = u \prod_{i=1}^{n} t_i^{e_i}$$

for non-negative integers e_i and a unit $u \,\epsilon\, O_p$ such that the t_i appearing in the product are linearly independent elements of the cotangent space $\frac{m}{m^2}$.

We are going to use this assumption to define particularly useful sets of local uniformizing parameters in the local rings O_p.

8.3 Global Blow-ups

Before returning to the local situation, we make a remark on the nature of the blow ups of the base scheme and the total space. If we need to blow up the base B, either the blow-ups will be contained in one affine neighborhood, or we will perform a blow up at a sheaf of ideals defined by the intersection of two or more components of the discriminant locus. In this way the blow-ups can be defined globally. The blow ups to desingularize the total space X are also canonical enough to be defined globally, since we can define these blow ups geometrically.

For example, we frequently put ourselves in a situation where a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{100}$$

has coefficients in some ring R, and $t = 0$ for some $t \epsilon R$ defines a component of the discriminant locus, and furthermore all singularities of X contained in the subscheme defined by $t = 0$ are at $x = y = 0$. In this situation we blow up at the ideal (t, x, y). Note that this ideal is geometrically defined by a component of the discriminant locus and by the coordinates of the singularities.

All blow ups of the base B and of the total space X will be defined geometrically, and this allows us to work in one affine piece at a time. Since we are interested in the regularity of X and the special fibers over closed points of B, the effect of these blow ups will be analyzed in the local rings $O \epsilon B$.

8.4 Obtaining a DVR

Starting with a closed point $p \epsilon B$ which contains the discriminant locus, we work in the local ring O_p. Suppose t_i, $i = 1, ...k$ define the reduced discriminant locus. If we assume that the discriminant locus has normal crossings, the t_i can be completed to a set of uniformizing parameters for the local ring. If the local ring O_p has maximal ideal m, the t_i, $i = 1, ...n$ form a basis for the vector space $\frac{m}{m^2}$. Write q for the characteristic of $\frac{m}{m^2}$; q may be positive or zero. Explicitly, we make a definition.

Definition 8.7 (Uniformizing Parameters)
Let E be an elliptic scheme defined by a Weierstrass equation f over a regular n dimensional local ring O with maximal ideal m. Suppose $t_1, t_2, ...t_k$ define the reduced discriminant locus in Spec(O), and further suppose that the discriminant locus has normal crossings. If $t_{k+1}, ...t_n$ extend $t_1, t_2, ...t_k$ to a system of uniformizing parameters for m, then we say the $\{t_i\}$ form a Discriminant Compatible Set of Uniformizing Parameters (for m with respect to E)

Next, for each t_i, we can localize O_p away from $t_i = 0$. This produces a DVR R_i with uniformizer t_i and residue field R_i/t_iR_i. This residue field may have either characteristic 0 or characteristic q and may or may not be perfect. We now give notation for these ideas.

Definition 8.8 (Localization Notation)
Let O be a regular n dimensional local ring with maximal ideal m. Suppose $\{t_i\}$ is a set of uniformizing parameters for m. Then for each i define a DVR R_i to be O localized away from the ideal (t_i). Define κ_i to be the residue field of R_i. Define v_i to be the valuation associated to the DVR R_i. Define H_i to be the hypersurface in Spec(O) defined by $t_i = 0$. If E is an elliptic scheme over O given by a Weierstrass equation f, define E_i to be the elliptic scheme over R_i defined by tensoring the coefficients of f by R_i.

We already know how to find a regular model for the elliptic scheme over a DVR by using the translations, charts, and blow ups in the previous sections. The special fiber $t_i = 0$ will be a Kodaira type, or one of the new special fibers described in Section 7.2. Thus, for each component $t_i = 0$ of the discrimant locus we can define its type.

Definition 8.9 *Let X be an elliptic scheme over B. Let D be a component of the discriminat locus defined locally at a point $p \epsilon B$ by $t_i = 0$. Let E_i be the elliptic scheme defined over the DVR R_i. Then the type of the component D is the reduction type of E_i as defined in section 4.*

Although the type of a component of the discriminant locus is defined locally, it is well defined.

The uniformizing parameters described are above are special since a subset of them describe the discriminant locus. In some of the arguments which follow, we will need to know when a given element of O_p defines a reduced subscheme of the zero locus of these t_i. Such an element will behave well in the blow up constructions which follow.

Definition 8.10 (A $\{t_i\}$ Normal Element)
Let O be a regular n dimensional local ring with maximal ideal m. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for m with respect to E. An element $a \,\epsilon\, O$ is a $\{t_i\}$ Normal Element if

$$a = u \prod_{i=1}^{n} t_i^{e_i}$$

for integers e_i and a unit $u \epsilon O$.

Clearly by definition the discriminant d is a $\{t_i\}$ Normal Element. We make one more definition which generalizes the notion of a minimal Weierstrass equation ([SIL 1]) over a DVR.

Definition 8.11 (Minimal Weierstrass Scheme)
Let X be a Weierstrass elliptic scheme over B. We call X a Minimal Weierstrass Elliptic Scheme if for each $p \epsilon B$ and each DVR R_i, the Weierstrass equation over R_i defining E_i is a minimal Weierstrass equation.

We now proceed to use the above notation and definitions to produce an analog of Weierstrass equations in chart form when working over our more general schemes.

8.5 Simultaneous Translations

In the previous sections we showed that for an elliptic scheme over a DVR R defined by a Weierstrass equation f, it was possible to perform R translations so that the valuations of the a_i fell into a pattern of one of the types on the appropriate chart, that is f could be translated to chart form. We would like to have a result for higher dimensional local rings.

Given a local ring O, and a Weierstrass equation with coefficients in O, we would like to perform O integral translations such that for each t_i in the set of uniformizing parameters, the Weierstrass equations over R_i is in chart form. Unfortunately this may not be possible without further assumptions. However we can always perform O integral translations such that for each t_i in the set of uniformizing parameters, the Weierstrass equations over R_i is at least in pre-chart form.

To specify the O translations we use the translations specified in the DVR case. For each R_i the extended Tate's algorithm guarantees us R_i translations that put the equation in pre-chart form, (and ultimately in chart form). We first note that these translations are only defined modulo t_i, so we are just searching for an element of O that maps to a specific element α of κ_i under the natural map

$$O \to R_i \to \kappa_i \tag{101}$$

Since there are translations in each "direction" defined by each of the t_i in the discriminant locus, we want to show that these translations are compatible with one another. We want

Proposition 8.12 (Multiple Pre-Chart Form)
Let E be a minimal Weierstrass elliptic scheme defined by a Weierstrass equation f over a regular n dimensional local ring O with maximal ideal m. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for m. Then there exists a single O translation for x, and a single O translation for y producing a translated Weierstrass equation f' such that for each $i \in \{1, 2...n\}$, the Weierstrass equation f' over R_i is in pre-chart form. After such an O translation we say f is <u>*In Multiple Pre-Chart Form*</u>.

The first step in proving Proposition 8.12 is to show that for a given i the R_i translations called for in Tate's algorithm can be lifted to O translations.

8.5.1 Integrality of Cubic Translations

The translations in Tate's algorithm that put a Weierstrass equation in pre-chart form involve translating a singularity of the cubic curve 2 to $(0,0)$ and translating a multiple root of a quadratic or cubic polynomial (7, 8, 9, 10) to 0. In this section we deal with the former case.

Proposition 8.13 (Cubic Translations are Integral)
Let E be an elliptic scheme defined by a Weierstrass equation f over a regular n dimensional local ring O with maximal ideal m. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for m. Fix an index $i = i_0$. If there is a singular point on the special fiber of E_i, and there are R_i translations that translate it to $(0,0)$, then there are also such R_i translations that can be lifted to O translations.

Given our elliptic scheme defined by a Weierstrass equation f over O, we consider the Weierstrass equation over R_i that defines E_i. In case $v_i(d) = 0$ or the reduction type of f over R_i is $X1$ or $Y1$ if $char(\kappa) = 2$, or type $Z1$ if $char(\kappa) = 3$, we do not need to translate a singular point to $(0, 0)$.

If the reduction type is not one of the above few, Tate's algorithms does require a translation. This means there are elements $x_0, y_0 \,\epsilon\, R_i$ such that modulo t_i, the pair (x_0, y_0) defines the singular point of the cubic over κ_i. Let us refer to the following commutative diagram.

$$\begin{array}{ccccc} Frac(O) & \leftarrow & R_i & \leftarrow & O \\ & & \downarrow & & \downarrow \\ & & \kappa_i & \leftarrow & O/(t_i) \end{array} \qquad (102)$$

Here $O/(t_i)$ is just O modulo the ideal (t_i). Considering x_0 and y_0 as elements of $\kappa_i = Frac(O/(t_i))$, we will show that $x_0, y_0 \epsilon O/(t_i)$.

A singular point on a cubic given by a Weierstrass equation 2 f over a field must satisfy $f = 0$, $\frac{df}{dx} = 0$, and $\frac{df}{dy} = 0$. We obtain three relations in $\kappa_i = Frac(O/(t_i))$ The relations are:

$$y_0^2 + a_1 x_0 y_0 + a_3 y_0 = x_0^3 + a_2 x_0^2 + a_4 x_0 + a_6 \qquad (103)$$

$$a_1 y_0 = 3x_0^2 + 2a_2 x_0 + a_4 \qquad (104)$$

$$2y_0 = -a_1 x_0 - a_3 \qquad (105)$$

Construct one more by taking Equation 105 and subtracting y_0 from both sides.

$$y_0 = -a_1 x_0 - a_3 - y_0 \qquad (106)$$

Notice that although x_0, y_0 $\epsilon Frac(O/(t_i))$, we have the $\{a_i\} \epsilon\, O/(t_i)$. We are going to show that x_0, y_0 $\epsilon O/(t_i)$. To do this we will assume the contrary and write x_0 and y_0 as reduced fractions with numerator and denominator in $O/(t_i)$. Let p be a prime element in O dividing the denominator of one of x_0 or y_0. We define a valuation function v on $Frac(O/(t_i))$ by setting $v(p) = 1$ and extending by the usual non-archimedian valuation properties. We aim to show $v(x_0) \geq 0$ and $v(y_0) \geq 0$.

Now apply the valuation function to both sides of equations 103, 106, 104, and 105 and use the facts that $v(a_i) \geq 0$, and $v(a+b)= min(v(a), v(b))$ when $v(a) \neq v(b)$. We obtain four equations:

$$v(y_0) + v(y_0 + a_1x_0 + a_3) = 3v(x_0). \tag{107}$$

$$v(y_0) = v(a_1x_0 + y_0 + a_3). \tag{108}$$

$$v(a_1) + v(y_0) = 2v(x_0). \tag{109}$$

$$v(y_0) = v(a_1) + v(x_0). \tag{110}$$

Equation 109 only holds provided $v(3) = 0$, and equation 110 holds provided $v(2) = 0$.

The first two equations 107 and 108 clearly imply

$$2v(y_0) = 3v(x_0). \tag{111}$$

Since p can not divide but 2 and 3, we can not have both $v(2) > 0$ and $v(3) > 0$, so first assume $v(2) = 0$. Then equations 111 and 110 imply

$$2v(a_1) = v(x_0). \tag{112}$$

Assuming instead $v(3) = 0$, we use equations 111 and 109 to conclude the same relation 112. In either case, since $v(a_1) \geq 0$, indeed $v(x_0) \geq 0$ and thus also $v(y_0) \geq 0$.

Thus p does not divide the denominator of either x_0 or y_0. This argument is valid for any prime element p of $O/(t_i)$, thus we conclude that that $x_0, y_0 \,\epsilon\, O/(t_i)$. Now simply choose representatives for x_0, and y_0 in O to conclude Proposition 8.13. Of course x_0, y_0 can always be changed by a multiple of t_i.

We also note that the argument simplifies greatly if we assume that $\frac{1}{2}\epsilon O$ and we work with the simpler Weierstrass equation.

$$y^2 = f = x^3 + a_2x^2 + a_4x + a_6 \tag{113}$$

In this case any singularity in κ_i must be at $(x_i, 0)$ where $x_i \epsilon R_i$ defines a multiple root of f over κ_i. Because f is monic, and each $a_i \epsilon O$, we must also have $x_i \epsilon O$.

8.5.2 Integrality of Remaining Translations

In this section we examine the other translations in Tate's algorithm that put a Weierstrass equation in pre-chart form.

In the stages of Tate's algorithm after the singularity (x, y) has been translated to $(0, 0)$, we have seen that there are other translations to be made. We will ignore the translations involving the K_n , I_n and I_n^* families because they are not needed to achieve pre-chart form. I recall the polynomials that specify the translations to achieve pre-chart form.

$$Y^2 + a_1 XY - a_2 X^2 \tag{114}$$

$$Y^2 + \frac{a_3}{\pi} Y - \frac{a_6}{\pi^2} \tag{115}$$

$$X^3 + \frac{a_2}{\pi} X^2 + \frac{a_4}{\pi^2} X + \frac{a_6}{\pi^3} \tag{116}$$

$$Y^2 + \frac{a_3}{\pi^2} Y - \frac{a_6}{\pi^4} \tag{117}$$

Tate's algorithm provides an R_i translation that moves any multiple root of one of the above polynomials to 0.

Again our arguments simplify if we assume that $\frac{1}{2} \epsilon O$ and we work with the simpler Weierstrass equation.

$$y^2 = x^3 + a_2 x^2 + a_4 x + a_6 \tag{118}$$

because we need only consider equation 116 in this case.

Proposition 8.14 (Polynomial Translations)
Let E be an elliptic scheme defined by a Weierstrass equation f over a regular n dimensional local ring O with maximal ideal m. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for m. Fix an index $i = i_0$. If an R_i translation is needed to translate a multiple root of one of the polynomials 114, 115, 116, or 117 defined over κ_i, then this R_i translation can be lifted to an O translation.

Suppose that $\alpha \epsilon R_i$ defines a multiple root of one of the above polynomials in κ_i. Consider $\alpha \epsilon Frac(O/(t_i))$. Because all of the coefficients are elements of O, and the polynomials are monic, we must have $\alpha \epsilon O/(t_i)$. Now simply choose a representative for α in O to conclude Proposition 8.13. Of course α can always be changed by a multiple of t_i.

There is but one translation we have not considered. Suppose that the characteristic of κ is 2, and that f has a singularity that can not be translated to $(0,0)$. Then we have either type $X1$ or $Y1$. In order to put the $Y1$ type in chart form, we must translate any double root of $x^2 + a_4$ to 0. Because this polynomial is monic, such an R_i translation can also be lifted to an O translation.

The two propositions 8.13 and 8.14 together show that for a given DVR R_i, we make choose O translations to put the Weierstrass equation over R_i in chart form. If we can do this compatably for each i, we can prove proposition 8.12.

8.5.3 Multiple Translation Compatibility

Let t_i be a discriminant compatible set of uniformizing parameters at p. We now fit together the translations required to put the elliptic schemes over the R_i in pre-chart form. This algorithm, which will prove of Proposition 8.12 can also be viewed as a higher dimensional version of Tate's algorithm.

First, for each index i such that there is a singularity on the scheme over R_i that can be translated to $(0,0)$, choose elements x_i and $y_i \epsilon O$ that define the O translations. Then for each fiber over the base with $t_i = 0$, the node or cusp is at $x = x_i, y = y_i$, and the x_i, and y_i are defined modulo modulo t_i.

Consider a fiber of the base where two or more of the $t_i = 0$. At such a point the curve can still have at most one node or cusp. This means that for any set of indices $\{i_1, ..i_k\}$ we have

$$x_1 = x_2 = ...x_k \, mod \, (t_1, ..t_k) \quad (119)$$

$$y_1 = y_2 = ...y_k \, mod \, (t_1, ..t_k). \quad (120)$$

Now use the following proposition.

Proposition 8.15 (Local Ring Patching)
Let O be a local ring. Let t_i be a finite number of elements in the maximal ideal indexed by $i \epsilon I$. Let $x_i \epsilon O$ such that for any set of indices $\{i_1, ..i_k\} \subset I$ we have

$$x_1 = x_2 = ...x_k \, mod \, (t_1, ..t_k)$$

Then there exists a single element $x \epsilon O$ such that $x = x_i$ modulo t_i for all indexes i.

The proof is easy induction. In our case it is thus possible to choose one element x_0 and one element y_0 of O such that $x = x_i$ modulo t_i and $y = y_i$ modulo t_i for all indexes i.

For example, if there were just two translations x_1 and x_2 required, we have $x_1 = x_2 \bmod (t_1, t_2)$ so that $x_1 - x_2 = t_1 a + t_2 b$. Now set $x_0 = x_1 - t_1 a$, and check that $x_0 = x_1 \bmod t_1$ and $x_0 = x_2 \bmod t_2$.

Step 1 in the procedure for proposition 8.12 is to take the Weierstrass equation over O and translate x by x_0 and y by y_0.

Step 2: If one or more the reduction types over some R_i's is of type $Y1$, choose one x_0 such that x_0 is the double root of the polynomial $x^2 + a_4$ for each such index i. This may be done for multiple indices if needed by proposition 8.15. If for one or more other indices j a singularity of the curve E_j over R_j has been translated to $(0, 0)$ in the previous step, we will already have $a_4 = 0 \bmod t_j$, so such an x_0 can be chosen as not to interfere with the previous steps' translations.

Step 3: For each index i such that the polynomial 114: $Y^2 + a_1 XY - a_2 X^2$ has a double root over κ_i, we may choose a single β_0 (again by proposition 8.15). such that for each such index the polynomial factors in κ_i as

$$(Y - \beta_i X)^2 \tag{121}$$

and $\beta_0 = \beta_i \bmod t_i$ for each such index. Now translate the Weierstrass equation over O, translate by $y \to y + \beta_0 x$.

At this point any further translations of x or y will be by a multiple of $t_{i_1} ... t_{i_k}$ where $i_1 ... i_k$ are the indices for which translations have been made.

Step 4: By similar arguments any double roots of the polynomial 115 over some of the κ_i's can be translated to zero by a single O translation of Y.

For example, if we need to translate the zero's of polynomial 115 to 0 for two indices t_1 and t_2, we would find one element $x_0 \epsilon O$ such that the roots of

$$Y^2 + \frac{a_3}{t_1 t_2} Y - \frac{a_6}{t_1^2 t_2^2} \tag{122}$$

over κ_i and k_2 were equal to x_0 modulo t_1 and t_2 respectively. Then we translate by $Y' = Y - t_1 t_2 x_0$.

Step 5: Translate X by by a single element $x_0 \epsilon O$ such that any multiple roots of the polynomial 116 over some of the κ_i are zero.

Step 6: Translate Y by by a single element $y_0 \epsilon O$ such that any multiple roots of the polynomial 117 over some of the κ_i are zero.

Step 7: If there exists any one t_i such that

$$(t_i)^j \mid a_j \tag{123}$$

for $j = 1, 2, 3, 4, 6$, restart the procedure with $a'_j = \frac{a_j}{t_i^j}$. If this happens, we did not start with a minimal Weierstrass elliptic scheme, as required in the theorem. Frequently we do not know if our Weierstrass elliptic scheme is minimal, so we may relax this condition of 8.12 and consult the following section 8.6.

This concludes the sketch of the proof of Proposition 8.12. A Weierstrass equation can be simultaneously translated for all indices i into pre-chart form.

8.6 Defining a Minimal Scheme

In this section we discuss how to replace a Weierstrass elliptic scheme with a minimal Weierstrass elliptic scheme. By definition a Weierstrass elliptic scheme is not minimal if for some point $p \epsilon B$, and some DVR R_i at p, the corresponding Weierstrass equation defining E_i over R_i is not minimal.

Starting with a non-minimal scheme X over B we will construct the minimal one. Suppose X is defined by f at p and is not minimal at p. Following the proof of proposition 8.12, there is some element $t_i \epsilon O$ that defines a hypersurface H_i in the discriminant divisor of B such that at p, we have the relation 123. We note then that X is not minimal over every point of this component H_i of the discriminant divisor. Let $U \subset B$ be an affine open containing p over which X is defined by the Weierstrass equation f.

Let f' be a Weierstrass equation with coefficients

$$a'_j = \frac{a_j}{t_i^j} \tag{124}$$

and consider the subscheme X' of $P^2(U)$ cut out by f'. There is a natural rational map $X \to X'$ given by the ring maps

$$\begin{aligned} P^2(U)/(f) &\to P^2(U)/(f') \\ x' &\mapsto x/{t_i}^2 \\ y' &\mapsto y/{t_i}^3 \end{aligned} \tag{125}$$

This morphism is an isomorphism on the open subscheme $t_i = 0$. Unfortunately, this map does not extend to a morphism $X \to X'$, or $X' \to X$. See [HART] pp. 80 for an exercise discussing when a map of graded rings defines a map of $Proj$ Schemes.

We define a whole more minimal elliptic scheme X' over B by performing this construction for all affine open subsets of B that contain H_i. It is easy to construct the compatibility maps for points p contained in more than one affine open. Thus we can patch these affine schemes to obtain a scheme over B via this construction. We repeat the process for other components H_i of the discriminant divisor over which the elliptic scheme is not minimal. Because B is Noetherian and each component of the discriminant divisor has finite multiplicity, this process will eventually terminate, and the resulting scheme X' is still birational to X and we have the morphism

$$X \to X'. \tag{126}$$

We remark that the discriminant divisor on B of X' is strictly less than the discriminant divisor on B of X, and it is possible that some components completely disappear.

The above discussion plays a role for the following reason: We will take a Weierstrass elliptic scheme over some base B, blow up the base via $B' \to B$, and pull back X to B'. Even if the original Weierstrass elliptic scheme is minimal over B, the pulled back Weierstrass elliptic scheme may not be minimal over B'. Each time we blow up the base we may invoke this construction. Indeed, in section 14, we repeatedly blow up the base and use combinatorial arguments to limit the types of collisions in a settled elliptic scheme.

8.7 Translation Necessity

In this section we make a digression to discuss the necessity of translating Weierstrass equations. It Tate's original exposition [TA], not all of the translations that we require were made before blowing up.

Let E be a Weierstrass elliptic scheme E over a DVR R with residue field κ. If there is a singular point in the special fiber we could just blow up at the singular point of the surface, without ever considering a translation. Of course we can only do this if the singular point is κ rational.

If we can choose $x_0, y_0 \,\epsilon R$, such that the singular point in the special fiber is given by (x_0, y_0) modulo κ, then the singular point is rational. Equivalently there exists a section s

$$\begin{array}{c} E \\ \uparrow \\ Spec(R) \end{array}$$

such that the singularities in the special fiber lie in the image of s.

Now consider a Weierstrass elliptic scheme E over an n dimensional regular local ring O. Let t_i be a discriminant compatible set of uniformizing parameters for O. Proposition 8.12 states that there is a Weierstrass equation defining E that is in multiple pre-chart form. In terms of sections, this means that there is a section s

$$\begin{array}{c} E \\ \uparrow \\ Spec(O) \end{array}$$

such that for each component H_i of the discriminant locus, such that the singularity on the special fiber of E_i is k_i rational, all singularities of E over H_i are contained in the the image of s.

The conditions of pre-chart and chart form for a Weierstrass elliptic scheme can both be rephrased in terms of existence of sections. This serves to help explain the geometry of the schemes. However, it is easier to define the concepts of pre-settled and settled by using Weierstrass equations that are in chart from. Furthermore, the blow up computations become more difficult if the singularities are not translated to specific points.

8.8 Further Steps

Let us look more closely at the ideals of the blow-ups in the schemes over the DVRs. Since the ideals were defined only by using x, y and t_i with t_i a uniformizer in the DVR, these ideals can all be pushed forward to ideals defining subschemes of E over the base O_p. Thus we take the blow ups prescribed by Tate's algorithm and perform them on the larger scheme over the higher dimensional rings O_p. As remarked in section 8.3, these blow ups are local versions of the global blow ups described by subschemes of X or B, or equivalently by sheaves of ideals on X or B.

The regularity of the total space over the DVR guarantees that over at a generic point $p \epsilon X$ in a fiber with $t_i = 0$, the scheme will be regular. However, we may not have regularity of every point in this fiber, especially over a non regular point in the discriminant locus. For example, we must check regularity over ideals such as (t_1, t_2), where both t_1 and t_2 define components of the discriminant locus. The fibers over such points of the base are called collisions, and we will need to explore criteria for regularity at collisions.

To provide such a criteria, we introduce the concepts of *pre-settled* and *settled*. In the pre-settled case we will be able to extend the arguments of this section to prove that we can achieve multiple chart form, not just multiple pre-chart form. In the pre-settled and settled case we severely limit the possible types collisions, and can algorithmically further limit certain collisions. This will put us in the situation where the blow ups of Tate's algorithm are sufficient to guarantee regularity over all points of the base including points corresponding to collisions. Finally, we can give a good description of the double or higher dimensional collision special fibers.

9 Blowing up the Base

9.1 Elliptic Scheme Pullback

Given a Weierstrass elliptic scheme X over B we would like to desingularize X by performing the blow ups specified by Tate's algorithm. To this end, we have shown by proposition 8.12 that over a local ring, X can be cut out by a Weierstrass equation in pre-chart form. Our first problem is that the Weierstrass equation may not be completely in chart from, and more seriously, if we do follow the blow ups given by Tate's algorithm producing $X' \to X$, we are not guaranteed that X' is regular over points of B where the discriminant divisor has multiple components.

Our strategy will be to blow up B and pull back X to a new base as follows.

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ B' & \to & B \end{array} \tag{127}$$

We thus hope to overcome the second problem described above.

In this construction, the pullback of the discriminant divisor of X of B will be the discriminant divisor of X' of B'. The new discriminant divisor may have a new component. Unfortunately, the reduction type of the new component is not always predictable based on the reduction types of the original components.

We would like to be able to predict the type of the new components, and we will provide criteria in sections 10 and 12 for when the new components are indeed predictable. In these sections we introduce the notions of "settled", and "pre-settled". Secondly, if such new components are predictable, in section 14 we will algorithmically reduce to cases where performing the blow ups given by Tate's algorithm does produce a regular scheme. Before attempting this resolution, we should also make sure that our Weierstrass elliptic scheme is locally in full chart form.

An alternate strategy is to produce a well defined J morphism on the base. This has the disadvantage of only working when $\frac{1}{6}$ is in the local rings. However, combined with the assumption 8.5 that the discriminant locus has normal crossings, the existence of such a morphism is strong enough to allow us to predict the types of the new discriminant components. This technique and the relationship with the notions of settled and pre-settled is discussed in section 15.

We next review blow up computations using coordinates and computations involving the pullback of a divisor.

9.2 Review of Blow-ups

Here we review the definition and interpretation of blow-ups, and particularly focus on blowing up a local ring. We have indeed already computed blow ups in section 6.

A blow is a type of morphism of schemes $X' \to X$ that attempts to desingularize X. Geometrically, blowing up a scheme at a point separates tangent vectors at the point. One classical definition is given in [HAR]. It presents a blow up as a quasi projective subvariety of $A^n \times P^n$.

Definition 9.1 *Let A be an affine subvariety of $A^n = Spec(k[x_1, ...x_N])$ passing through the point p given by $x_1 = ... = x_n = 0$. Then the Blow up of A at p is the graph of the birational map to $P^n = Proj[y_1, ...y_n]$ given by $y_i = x_i$.*

A more general and technical definition is that a blow up is the *Proj* of a sheaf of graded rings [HART]. The graded rings in the construction are called blow up algebras.

Definition 9.2 *Let X be a scheme and I be a coherent sheaf of ideals on X. The Blow up of X at I is the scheme obtained by patching together the following schemes for ecah affine open $U = Spec(R)$ in X.*

$$Proj(R + It + I^2t^2 + ...)$$

Here $I \subset R$ is the ideal $I(U)$ from the sheaf I, and the ring $(R+It+I^2t^2+...)$ is given a grading by setting degree(t)=1.

When computing in coordinates, we need to understand this latter definition. A discussion of how the *Proj* of a graded ring R can be patched together from affine schemes can be found in [EIS-HAR]. I mention that the rings in the construction are the zero'th graded pieces of localizations of R. A detailed example of blowing ups a hypersurface in projective space using coordinates is presented in [SIL 2].

9.2.1 Blowing up a Local Ring

Now we discuss the local effect of blowing up a regular Noetherian scheme X at a regular subscheme S. We assume that S is reduced, regular, and irreducible. So start with a local ring O of dimension n and and ideal I cutting out S. With these assumptions, R/I is also a local Noetherian regular ring, and thus a complete intersection ring. By a theorem in commutative algebra, [MAT] 21.2, I must be generated by an R-regular sequence. So write

$$I = (t_1, ...t_k). \tag{128}$$

By another theorem in commutative algebra, [MAT] 17.4, the $\{t_i\}$ can be extended to a set of uniformizing parameters for O. Then the blow up of X along S is given by

$$Proj(O[a_1, ...a_k]/(a_i t_j - a_j t_i) \tag{129}$$

Where the a_i have grading 1. The first coordinate patch of the *Proj* is thus given as

$$Spec(O[a_2, ...a_k]/(t_i = t_1 a_i)) \tag{130}$$

Where the a_i are dehomogenized coordinates. Note that the above ring is not local, but at a given point it is easy to find uniformizing parameters. For example at the local ring of the point $a_2 = ... = a_k = 0$ a set of uniformizing parameters is

$$\{t_1, a_2, ...a_k\} \tag{131}$$

We now know how to compute blow ups locally in coordinates. Secondly, it is easy to apply the Jacobian criterion mentioned in section 6.3 to check that this last scheme is regular. We obtain the following result

Proposition 9.3 *Let X be a Noetherian regular scheme. Let S be a regular subscheme of X. Then the blow up of X along S is again regular.*

9.3 The Exceptional Divisor

In this section we continue to consider the blow up of a local ring, and consider the preimage in B of a divisor on O.

Let X be a Weierstrass elliptic scheme over a local ring O with maximal ideal m. Assume $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for m. We blow up O at the subscheme defined by the intersection of two or more of the hypersurfaces H_i. For simplicity reorder the indices and assume that we are blowing up at the intersection of the first l hypersurfaces H_i.

$$B \rightarrow Spec(O) \tag{132}$$

After pulling back X to the new base B, we still have a Weierstrass elliptic scheme, and any divisor on O pulls back to a divisor on B. We can then consider the component of this divisor that is supported in the exceptional divisor of the blow up. To calculate its degree we consider the valuation at the DVR defined by the exceptional divisor of the blow up. We now define these terms.

Definition 9.4 (Exceptional Divisor)
Let X be a Weierstrass elliptic scheme defined over a regular n dimensional local ring O with maximal ideal m. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for m. Let $B \rightarrow Spec(O)$ be the blowing up at the subscheme S defined by the ideal $(t_1, ...t_l)$. Define the Exceptional Divisor of the blow up to be the inverse image S. In each of the local rings of B we can localize at the exceptional divisor to obtain a DVR R_e. The valuation v_e associated to R_e is called the Exceptional Valuation.

Although O is a local ring the blowing up of it B will not be. So we consider O', a local ring of B. We note that we may produce new uniformizing parameters for O' from the original $\{t_i\}$ in the following sense.

Proposition 9.5 (New Uniformizers)
Let X be a Weierstrass elliptic scheme defined over a regular n dimensional local ring O with maximal ideal m. Suppose $\{t_i\}$ is a discriminant compatible

set of uniformizing parameters for m. Let $B \to Spec(O)$ be the blowing up at the subscheme S defined by the ideal $(t_1, ...t_l)$. Let $t_e = 0$ define the exceptional divisor of the blow up. Let O'_p be a local ring of B. Let u_i be the subset of the $\{t_i\}$ such that the hypersurface H_i passes through the point p. Then the set $\{t_e, u_{i_1}, ..u_{i_l}\}$ extends to a discriminant compatible set of uniformizing parameters for O'_p.

Thus the uniformizing parameters to differ at different points on the exceptional divisor. However t_e belongs to this set for every point on the exceptional divisor. We also can pull back any element $x \epsilon O$ to an element in O'. We would like to be able to compute the exceptional valuation of x in terms of the various valuations v_i of x in O.

Proposition 9.6 (Computing the Exceptional Valuation)
Let O be a regular n dimensional local ring O with maximal ideal m. Suppose $\{t_i\}$ is a set of uniformizing parameters for m. Let $B \to Spec(O)$ be the blowing up at the ideal $(t_1, ...t_l)$. In any local ring O' of B, let v_e be the exceptional valuation. Let $x \epsilon O$, and let x' be the image in O'. Then

$$v_e(x) = max\,(n \,|\, x \,\epsilon\, I^n) \tag{133}$$

The proof of this follows directly from the definition of a blow-up. In particular, before passing to a more minimal scheme by replacing the Weierstrass equation with a minimal one as in section 8.6, we apply the relation 133 to $a_1, ..., a_6$ and to d, the discriminant divisor itself.

9.4 Predicting the Exceptional Divisor

As we blow up the base at ideals such as $(t_1, ...t_l)$, we would like to predict the reduction type of the exceptional divisor. Ideally, we would like to obtain a Weierstrass equation already in pre-chart or chart form given the fact that we started with a Weierstrass equation in pre-chart or chart form. Since the reduction types are mainly determined by the valuations of the a_i, and d, we focus on these.

By using equation 133 above, we see that if the a_i, or d are $\{t_i\}$ normal elements as defined in 8.7, we have the following relations.

$$a_i = u \prod t_k^{e_k} \tag{134}$$

$$d = u \prod t_k^{f_k} \tag{135}$$

with u a unit, and we blow up at the ideal $I = (t_1, ...t_l)$ then the exceptional valuations of the a_i's and of d are just the sum of the valuations $v_k(a_i)$ or $v_k(d)$ a in the DVRs R_k. That is

$$v(a_i) = \sum_{k=1}^{l} e_k \tag{136}$$

$$v(d) = \sum_{k=1}^{l} f_k \tag{137}$$

Although we assume that the discriminant divisor on the base has normal crossings,and thus that d is a $\{t_i\}$ normal element, we can not do this for the a_i. The a_i are not translation invariant, and we do need translations to guarantee chart form for the Weierstrass equations. Of course c_4, c_6, and d are translation invariant.

In general we can not predict the type of the exceptional divisor, and we will need to make assumptions that allow us to do so. Away from characteristic 2 and 3, the assumptions that d has normal crossings and the J morphism was well defined would be sufficient to allow us to predict the type of the exceptional divisor. Unfortunately, in the cases where the local ring O_p has residue characteristic 2 or 3, the reduction type of the exceptional divisor is not only a function of the reduction types along the hypersurfaces $\{t_i = 0\}$ participating in the collision, but rather depends on the exact valuations and values of the a_i's. This is because after each blow up, we may need further translations to ensure that our Weierstrass equations are in pre-chart or chart form.

For example, even if d has normal crossings and J is well defined, the discriminant divisor may have intersecting components of Type II and Type

III. The exceptional divisor of a blow up at the intersection could have many different reduction types, depending on the a_i's. This contrasts with the case $char \neq 2, 3$ where a Type II and Type III can not even collide provided the discriminant locus has normal crossings.

We explore the ramifications of a J morphism assumption in section 15. There we relate them to the more general concepts of settled and pre-settled.

10 A Pre-Settled Base

10.1 Grouping the Reduction Types

10.1.1 Motivation

As defined in 8.9, each component of the discriminant divisor has a type. The discriminant divisor may not be irreducible, so we make a definition.

Definition 10.1 *Let X be a Weierstrass elliptic scheme defind over B. Let D be the discriminant divisor on B. Fix $p \epsilon D$, and let $H_i = 0$ be the components of D passing through p. If the components H_i have reduction types T_i, we define say that the types T_i Collide at p, and that p is a Collision Point of B*

The valuation of the discriminant will be an important tool in controlling collisions. Our assumption that d has normal crossings allows us to use relation 137 to conclude that the valuation of the discriminant is additive in blow ups. However, this assumption is not sufficient to control collisions. One problem is that a given reduction type does not have a specified valuation of the discriminant. For example, if the characteristic of κ is 2 a type II, need not have $v(d) = 2$.

This chapter will divide the reduction types into various groups. The notion of pre-settled will be introduced as an assumption which eliminates collisions among reduction types coming from different groups. This is the first step in controlling which collisions are ultimately allowed. We define the groups differently at points P depending on whether or not the residue characteristic of O_p is 2.

10.1.2 Characteristic $\neq 2$

Consider a Weierstrass elliptic scheme defined over a local ring O_p of residue characteristic $\neq 2$. Let $\{t_i\}$, be a discriminant compatible set of uniformizing parameters for O. Then the residue characteristic of each of the R_i's is also different from 2. We will define three groups of reduction types, and each E_i may belong to one or more groups.

Examining chart 4.9, we see that for types I_n, and I_n^* the valuation of a_2 is required to be 0 and 1, respectively. We will put the types I_n, and I_n^* in the first group. I remark that given the existence of a J morphism one could conclude that $J = \infty$ for a reduction type if and only if the type was in the first group.

Now looking at the other types on the chart 4.9, we see that either a_4, or a_6 is set to a precise valuation. In fact we can separate the reduction types depending on whether the valuations of a_4^3 or a_6^2 is minimal. Put a reduction type in the second group if $v(a_4^3) \leq v(a_6^2)$, and put a reduction type in the third group if $v(a_4^3) \geq v(a_6^2)$. A reduction type I_0^* can be both the second and third group if $v(a_4^3) = v(a_6^2)$.

The reason to define the groups this way is the following observation. Suppose that at p, only types of the first group collide, and we blow up the base at p. Suppose also that a_2 was a $\{t_i\}$ normal element. Then in the DVR R_e of the exceptional divisor the valuation of a_2 would be minimal among a_2^6, a_4^3, and a_6^2. Consulting chart 4.9, we conclude reduction type of the exceptional divisor would also be in the first group.

The same argument works for the second and third groups provided a_4 or a_6 is a $\{t_i\}$ normal element.

10.1.3 Characteristic 2

Similar to the above case, we consider a Weierstrass elliptic scheme defined over a local ring O_p, but in this case suppose O_p has residue characteristic 2. Let $\{t_i\}$, be a discriminant compatible set of uniformizing parameters for O. Then at least one of the R_i's has residue characteristic 2. In this case we define five groups of reduction types.

As in the characteristic $\neq 2$ case, we see on the chart 4.2 that most of the reduction types require one of the a_i, to be set to a precise valuation.

Motivated by this chart, we group the reduction types based on which of the valuations of the following 5 monomials is minimal

$$a_1^{12} \quad a_2^6 \quad a_3^4 \quad a_4^3 \quad a_6^2 \,. \tag{138}$$

We form these groups for the same reason as in the above case. If all reduction types in a collision belong to the same group, then the reduction type of the exceptional divisor of a blow up is again in the same group. In this case we still need the assumption that the appropriate a_i is a $\{t_i\}$ normal element.

10.2 Definition of the Groups

10.2.1 Characteristic $\neq 2$

Here we make a formal definition which states which groups a given reduction types belongs to. We assume the local ring is not of characteristic 2.

Definition 10.2 (Groups in Char $\neq 2$)
Let E be an elliptic scheme defined over a local ring of residue characteristic $\neq 2$. For a discriminant compatible set of uniformizing parameters $\{t_i\}$, fix one E_i over the DVR R_i. Suppose E_i has reduction type T_i. We say T_i is in Group a_2 if it is type I_n or I_n^*. We say T_i is in Group a_4 if it is type III,III^*, or I_0^* with $v(a_4) = 2$. We say T_i is in Group a_6 if it is type $Z1$,II,IV,$Z2$,IV^*, II^*, or I_0^* with $v(a_6) = 3$.

Examining chart 4.9 we see that for each type in the first group $v(a_2) \in \{0, 1\}$. Each type in the second group has $v(a_4) \in \{0, 1, 2, 3\}$. Each type in the last group has $v(a_6) \in \{0, 1, 2, 3, 4, 5\}$. This the reason why the groups are so named.

10.2.2 Characteristic 2

Here we make a formal definition which states which groups a given reduction types belongs to in case the local ring is of characteristic 2. These groups are defined by which of the $a_i^{(\frac{12}{i})}$ is minimal.

Definition 10.3 (Groups in Char 2)
Let E be an elliptic scheme defined over a local ring of residue characteristic $\neq 2$. For a discriminant compatible set of uniformizing parameters $\{t_i\}$, fix one E_i over the DVR R_i. Suppose E_i has reduction type T_i. We say T_i is in Group a_1 if it is of type I_n. We say T_i is in Group a_2 if it is of type K_n,K'_n I_n, or $X1$ or $Y1$ with $v(a_2) = 0$. We say T_i is in Group a_3 if it is of type IV or IV^*. We say T_i is in Group a_4 if it is of type $X1$,III,$X2$,III^*, or I_0^* with $v(a_4) = 2$. We say T_i is in Group a_6 if it is of type $Y1$,II,$Y2$,$Y3$,$II*$, or $X1$ with $v(a_6) = 0$, or IV with $v(a_6) = 2$, or I_0^* or $X2$ with $v(a_6) = 3$, or IV^* with $v(a_6) = 4$.

As is evident by the definition, some of the types may belong to one or more of the a_i groups. Examining chart 4.2 we see that for each type in Group a_i, $v(a_i) \,\epsilon\, \{0, 1, ..i - 1\}$.

10.3 Straight and Residual Discriminant

To separate collisions between types in the groups described above, we will introduce the notion of a pre-settled point in the base. The notion of pre-settled is one that relates the valuations v_i of the discriminant d at each of the DVRs R_i, to that of the valuation $v_e(d)$ in the DVR R_e of the exceptional divisor defined by a blow up.

When working with a DVR of residue characteristic 0, the valuation of the discriminant is just determined by the reduction type. We do not make this assumption, and in order to separate the above defined groups, we define another measure. For any DVR we divide the valuation of the discriminant into two integers: A *straight* part and a *residual* part.

Definition 10.4 (Straight Part of d)
Let E be a Weierstrass elliptic scheme defined over a DVR R defined by a Weierstrass equation f in pre-chart form. Define the Straight Part of the

Discriminant, Straight(E), to be the minimum of the valuations of the elements

$$a_6^2 \quad a_4^3 \quad a_3^4 \quad a_2^6 \quad a_1^{12} \tag{139}$$

If we are dealing with the simpler Weierstrass equation.

$$y^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{140}$$

we can just omit a_1 and a_3 from the above definition.

Specifically, we have the following values of the straight part of the discriminant:

Reduction Type	*straight(E)*
$I_0, I_n, X1, Y1, Z1, K_n, K'_n$	0
II	2
III	3
$IV, Y2$	4
$I_0^*, I_n^*, X2, Z2, T_n$	6
$IV*, Y3$	8
$III*$	9
$II*$	10

(141)

We can also define a the residual part of the discriminant

Definition 10.5 (Residual Part of d)
Let E be a Weierstrass elliptic scheme defined over a DVR R. Then define the Residual Part of the Discriminant by

$$residual(E) = v(d) - straight(E) \tag{142}$$

When we have an elliptic scheme defined over a higher dimensional local ring O, we define these measures for each DVR R_i as follows.

Definition 10.6 (Indexed Straight and Residual Parts)
Let E be a Weierstrass elliptic scheme defined over a regular n dimensional

local ring O. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for O. Then for each i we consider E_i over R_i and define $straight_i(E) = straight(E_i)$, and $residual_i(E) = residual(E_i)$.

Clearly for all i,

$$residual_i(E) = v_i(d) - straight_i(E). \tag{143}$$

Since we may need to replace our Weierstrass elliptic scheme with a more minimal one as in section 8.6, we really consider the following sextuplet of valuations modulo the following vector of integers.

$$\begin{matrix} v(a_1) & v(a_2) & v(a_3) & v(a_4) & v(a_6) & v(d) \\ 1 & 2 & 3 & 4 & 6 & 12 \end{matrix} \tag{144}$$

Thus $straight(E)$ is also only defined modulo 12, and we choose for convenience to deal with $residual(E)$, which does not change when passing to a more minimal scheme.

10.4 Fault Integer

We defined the groups of reduction types as above to better predict the type of the exceptional divisor in a blow up. We remarked that if all reduction types in a collision belong to the same group then the type of the exceptional divisor is also in this group. This holds provided that the appropriate a_i is a $\{t_i\}$ normal element. This is the property that nice collisions should have.

With the definition of the straight and residual part of the discriminant above, these *nice* collisions have the property that after a blow up and before replacing the Weierstrass equation with a more minimal one, the following measures in R_e of the discriminant are related to the same measures in the DVRs R_i as follows.

$$v_e(d) = \sum_{j=1}^{k} v_j(d) \tag{145}$$

$$straight_e(E) = \sum_{j=1}^{k} straight_j(d) \tag{146}$$

$$residual_e(E) = \sum_{j=1}^{k} residual_j(d) \quad (147)$$

We now define a measure of the failure of the relations 146 and 147.

Definition 10.7 (Fault Integer)
Let E be a Weierstrass elliptic scheme defined over a regular n dimensional local ring O. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for O. Let $B \rightarrow Spec(O)$ be the blowing up at the ideal $(t_1, ...t_l)$, and let v_e be the valuation associated to exceptional divisor. Then we define The Blowup Fault at O to be the difference

$$Fault(E) = residual_e(E) - \sum_{j=1}^{l} residual_j(E) \quad (148)$$

Define the $PreFault_p(E)$ at p to be the maximum over all such blowups.

I remark that the maximum can be reached by blowing up at the closed point of O.

10.5 Spot Singularities

Let $X \rightarrow B$ be an elliptic scheme. The motivation for limiting collisions is so that we may desingularize X by applying Tate's algorithm for each component of the discriminant divisor. We have already mentioned that certain types of collisions may prevent this procedure from working. Furthermore, we may encounter other singularities even over smooth points of the discriminant locus.

Consider a point $p \epsilon B$ Such that exactly one component H of the discriminant locus passes through p, but the appropriate a_i for that group is not a $\{t_i\}$ normal element. Then after we apply Tates algorithm for the component H, the total space will be regular over the generic point of H but not over p. This can be verified by closely examining the last stage in Tate's algorithm where scheme is checked to be regular.

Such a point p can be called a *spot singularity*, because a singularity remains in the total space, yet p is not a collision point in the discriminant divisor.

Fortunately, the measure $Pre - Fault_p(d)$ defined above also detects this situation.

10.6 Defining Pre-Settled

A point $p\epsilon B$ is *pre-settled* if the residual part of the discriminant is additive. Equivalently there is no Pre-Fault at p.

Definition 10.8 (Pre-Settled)
Let $X \to B$ be a Weierstrass elliptic scheme defined over B. Let $p\epsilon B$. We say $X \to B$ is Pre-Settled at p if $PreFault_p(E) = 0$. An elliptic scheme is Pre-Settled if it is pre-settled at every point $p\epsilon B$.

This notion strong enough to allow only collisions of reduction types in the same group, to force the appropriate a_i to be $\{t_i\}$ normal elements, and to eliminate spot singularities. If we assume that our scheme is pre-settled, the reduction type of the exceptional divisor in a blow up will be more predictable.

10.7 A Rational Map

In this section we define local rational maps on the base. If our elliptic scheme is pre-settled this map will actually define a morphism of schemes. By considering the fibers of the morphism, we will have a more precise and general definition of the groups of reduction types. For each p in the base Let O_p be the local ring at p, and define a rational map from O_p to $P^4(O_p)$ as follows.

Definition 10.9 (Rational Map)
Let E be a Weierstrass elliptic scheme defined by a Weierstrass equation f

over a regular n dimensional local ring O_p. Suppose that f is in multiple pre-chart form. Define a rational map by

$$\psi(p) = (a_1^{12}, a_2^6, a_3^4, a_4^3, a_6^2) \tag{149}$$

Note that ψ is only defined locally and depends on the Weierstrass equation f. We compute the values of the rational map in the pre-settled case. Let $\{t_i\}$ be a set of discriminant compatible uniformizing parameters for O_p. For each index i cancel out the highest power of t_i in the definition of ψ. Notice that for each index i this highest power is equal to $straight_i(d)$, the straight part of the discriminant in the DVR R_i.

Supposing, that after this division, one of the components in the definition of ψ is a unit. Then the rational map is well defined at p, and the value $\psi(p)$ is just given by the reduction of each component to the residue field of O_p.

Now suppose that after that division every component of ψ is still in the maximal ideal of O_p. Then we consider blowing up the base at the maximal ideal of O_p, and choose any point q on the exceptional divisor. Let t be the uniformizer corresponding to the exceptional divisor. Note that ψ can be defined on the whole blow up of O_p.

When we consider ψ in one of the coordinate patches of the blow up, we see that one power of t can be factored out of the five-tuple defining ψ for each power of t_i that was previously factored out, plus at least one extra power of t.

This is because by equation 133 the valuation of an $x \epsilon O_p$ at t corresponds to the highest power of the maximal ideal that contains x.

In other words, before possibly replacing the Weierstrass equation with a more minimal one,

$$straight(E) > \sum_{j=1}^{k} straight_j(d) \tag{150}$$

Or equivalently

$$residual(E) < \sum_{j=1}^{k} residual_j(d). \tag{151}$$

This violates the pre-settled condition, and is thus a contradiction. We have proved the following proposition.

Proposition 10.10 (Rational Maps is a Morphism)
Let E be a Weierstrass elliptic scheme defined over B Suppose the elliptic scheme is pre-settled at $p\epsilon B$. Then the rational map ψ defined at p is a morphism.

Next we will consider fibers of the morphism to give a good description of what reduction types can collide in the pre-settled case.

10.8 Fibers of the Morphism

10.8.1 Characteristic $\neq 2$

In this case we can use the shorter form of the Weierstrass equation and also elliminate the components involving a_1 and a_3 in the definition of ψ.

$$\psi(p) = (a_2^6, a_4^3, a_6^2) \tag{152}$$

Since ψ is a morphism at least one component is a unit. We examine chart 4.9 and make a chart of which reduction types can occur at various fibers of the morphism ψ. This chart can also be regarded as a finer analysis of the groups defined above, because it also describes what reduction types can belong to more than one group. In this chart we omit mention of the Type I_0, and denote by an asterix an arbitrary unit.

Proposition 10.11 (Pre-Settled Collisions in Char $\neq 2$)
Let E be a pre-settled Weierstrass elliptic scheme given by a Weierstrass equation f over a local ring O_p of residue characteristic $\neq 2$. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O_p. Then if $\psi(p)$ is as in the first column of Chart 153, the only possible reduction types T_i over each R_i are as in the second column of Chart 153. Furthermore the the a_i mentioned in the third column must be $\{t_i\}$ normal elements in O.

$$\begin{array}{ccc} \underline{\psi(p)} & \underline{Types} & \underline{a_i\ normal} \\ (1,0,0) & I_n, I_n^* & a_2 \\ (0,1,0) & III, I_0^*, III^* & a_4 \\ (0,0,1) & Z1, II, IV, I_0^*, Z2, IV^*, II^* & a_6 \\ (0,*,*) & I_0^* & a_4, a_6 \\ (*,0,*) & I_0^* & a_2, a_6 \\ (*,*,0) & I_0^* & a_2, a_4 \\ (*,*,*) & I_0^* & a_2, a_4, a_6 \end{array} \qquad (153)$$

The I_0^* must have $v(a_4) = 2$ at values of ψ with the second component a unit, and must have $v(a_6) = 3$ at values of ψ with the third component a unit.

10.8.2 Characteristic 2

In this case we can use the full form of the definition of ψ. Since ψ is a morphism at least one component is a unit. We examine chart 4.2 and make a chart of which reduction types can occur at various values of ψ. In this chart we omit mention of the Type I_0, and denote by an asterisk an arbitrary unit. Note the residue field for some of the DVRs R_i obtained by localizing O_p may have residue characteristic 0, so we also refer to chart of section 4.6.

Proposition 10.12 (Pre-Settled Collisions in Char 2)
Let E be a pre-settled elliptic scheme given by a Weierstrass equation f over a local ring O_p of residue characteristic 2. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O_p. Then if $\psi(p)$ is as in the first column of Chart 154, the only possible reduction types T_i over each R_i are as in the second column of Chart 154. Furthermore the the a_i mentioned in the third column must be $\{t_i\}$ normal elements in O_p.

$\psi(p)$	$Types$	a_i $normal$
$(*,*,0,0,0)$	I_n	a_1, a_2
$(*,0,0,0,0)$	I_n	a_1
$(0,*,0,0,0)$	I_n, K_n, I_n^*, T_n	a_2
$(0,0,*,0,0)$	$IV, IV*$	a_3
$(0,0,0,*,0)$	$X1, III, I_0^*, X2, III^*$	a_4
$(0,*,0,*,0)$	$X1, I_0^*, X2$	a_2, a_4
$(0,0,0,0,*)$	$X1, Y1, II, IV, X2, Y2, I_0^*, IV^*, Y3, II^*$	a_6
$(0,0,*,0,*)$	IV, IV^*	a_3, a_6
$(0,0,0,*,*)$	$X1, I_0^*, X2$	a_4, a_6
$(0,0,0,*,*)$	I_0^*	a_2, a_6
$(0,*,0,*,*)$	$X1, I_0^*, X2$	a_2, a_4, a_6

(154)

We refer to the groups by which a_i's are required to be $\{t_i\}$ normal elements. The I_n type in the a_2 group must have residue characteristic 0. Furthermore, for each type in a group with a_i forced to be a $\{t_i\}$ normal element, the valuation of a_i must be the minimum allowed by the appropriate chart. For example, a type $X1$ in group a_2, a_4 must have $v(a_2) = 0$.

10.9 Groups are separated

The above analysis proves

Proposition 10.13 (Pre-Settled Separates Groups)
Let E be a pre-settled Weierstrass elliptic scheme defined over B with discriminant divisor D. If D has two or more component intersecting at a point $p \epsilon B$ then the reduction types of these components lie in the same group.

Thus if E ia a pre-settled Weierstrass elliptic scheme, and p is a point in the base B, by abuse of notation we can say that p itself belongs to a given group of reduction types.

We further note that a at point p that is not in group a_1 or a_2 none of the components of the discriminant divisor are a type in one of the infinite families of reduction types. We conclude that at such a point p, a Weierstrass equation in multiple pre-chart form is actually already in multiple chart form. If we were only interested in such points we could skip the following sections and jump directly to section 14. But to explore the behavior at points which involve the families of reduction types we must show first that we can translate to multiple chart form, and make the further assumption that the elliptic scheme is *settled*. This will allow us to define another rational map which will be a morphism and will further limit collision between types in the infinite families of reduction types.

11 Multiple Chart Form

We have already remarked that at a point p that is not in group a_1 or a_2 a Weierstrass equation in pre-chart form is already in chart form. In this section we deal with points of the base which do have reduction types in one of the infinite families and show that we can translate our Weierstrass equation into multiple chart form.

That is we aim to strengthen Proposition 8.12 to the following

Proposition 11.1 (Multiple Chart Form)
Let E be a pre-settled elliptic scheme defined by a Weierstrass equation f over a regular n dimensional local ring O with maximal ideal m. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for m. Then there exists a single O translation for x, and a single O translation for y producing a translated Weierstrass equation f' such that for each $i \in \{1, 2...n\}$, the Weierstrass equation f' over R_i is in chart form. After such an O translation we say f is <u>In Multiple Chart Form</u>.

To prove this we can already assume that f is in multiple pre-chart form. We then use the pre-settled hypothesis of the previous section to note that the reduction types involved must be members of the same group. We need only deal with the case where the group is of type a_1 or a_2. Because the analysis is simpler when the residue characteristic of O is not 2 we deal with this case first.

11.1 Characteristic $\neq 2$

In this case we can use the shorter form of the Weierstrass equation, and we do not need to consider any translations involving y. We further assume that we are dealing with reduction types which all belong to the a_2 group. These are just the I_n and I_n^* types. We let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters at p and assume f is in multiple pre-chart form. We have the situation where our Weierstrass equation is given by

$$y^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{155}$$

with a_2 a $\{t_i\}$ normal element. In fact if the reduction type is I_n over R_i then $v_i(a_2) = 0$, and if the reduction type is I_n^* over R_i then $v_i(a_2) = 1$.

We recall that to translate to chart form in the case of an elliptic scheme over a DVR R with uniformizer π we just needed to translate by the double roots of such polynomials as

$$a_2x^2 + \frac{a_4}{\pi}x + \frac{a_6}{\pi^2} \tag{156}$$

in the I_n case and

$$\frac{a_2}{\pi}x^2 + \frac{a_4}{\pi^3}x + \frac{a_6}{\pi^4} \tag{157}$$

in the I_n^* case. In the DVR case we did not need the fact that a_2 or $\frac{a_2}{\pi}$ was a unit in R, but we use this fact when dealing with a local ring O.

Let $\{r_i\}$ be a the subset of the $\{t_i\}$ such that f has a reduction type I_n over R_i, and $\{s_i\}$ be a the subset of the $\{t_i\}$ such that f has a reduction type I_n^* over R_i. Then for each r_i we have $v_i(a_2) = 0$, $v_i(a_4) \geq 1$, and $v_i(a_6) \geq 1$ and for each s_i we have $v_i(a_2) = 1$, $v_i(a_4) \geq 3$, and $v_i(a_6) \geq 4$. In the discussion which follows I will use only one index $r = r_i$ and one index $s = s_i$, but the argument is the same regardless of the number of indices present in the sets.

Since f is already in pre-chart form, we must ensure that all further translations of x are multiples of rs^2. We consider the polynomial

$$\frac{a_2}{s}x^2 + \frac{a_4}{rs^3}x + \frac{a_6}{r^2s^4} \tag{158}$$

We must translate x if it has a double root modulo r, modulo s, or both. because $\frac{a_2}{s}$ is a unit in O we know that any double root modulo r or modulo s can be lifted to an O integral element. Furthermore if there is a double root modulo r and modulo s, then it must agree modulo (r, s). This means we can find a single element x_0 such that x_0 reduces to the double root modulo r and modulo s. We then translate x by $-rs^2x_0$. Now all further translations of of x must be multiples of r^2s^3.

Assuming that the polynomial 158 has a double root modulo r and modulo

s, we would next examine

$$\frac{a_2}{s}x^2 + \frac{a_4}{r^2s^4}x + \frac{a_6}{r^4s^6} \tag{159}$$

for double roots. By the same argument there would be a single element x_0 such that x_0 reduces to the double root modulo r and modulo s. We then translate x by $-r^2s^3x_0$. Now all further translations of of x must be multiples of r^3s^4.

We repeat the process as long as the polynomial has a double root modulo r or modulo s. Of course once the polynomial does not have a double root we would no longer divide a_4 or a_6 by a higher power of r or s in the following stage. The same argument works when there are more than one element in the sets $\{r_i\}$ or $\{s_i\}$; we just divide a_2, a_4 and a_6 by the appropriate powers of r_i and s_i at each stage.

Thus we conclude that after a single O translation the Weierstrass equation f can be put into multiple chart form.

11.2 Characteristic 2

In this case we must use the full form of the Weierstrass equation, and we need to consider both translations involving x and y. We further assume that we are dealing with reduction types which all belong to either the a_1 group, the $a_1 - a_2$ group , or the a_2 group. The a_1 group, and the $a_1 - a_2$ group can be done at once. This group involves only I_n types. The a_2 group may contain also members of the K_n and I_n^* families.

We deal first with the a_1 group and the $a_1 - a_2$ group. This means we have a Weierstrass equation with a_1 a unit.

11.2.1 a_1 is a Unit

We recall that to translate to chart form in the case of an elliptic scheme over a DVR R with uniformizer π we just needed to translate by the possible singularities of quadratics of the form

$$y^2 + a_1xy + \frac{a_3}{\pi}y = a_2x^2 + \frac{a_4}{\pi}x + \frac{a_6}{\pi^2}. \tag{160}$$

Because $b_2 = a_1^2 + 4a_2$ is a unit, if the quadric is singular it consists of two intersecting lines. In fact we can solve for the coordinates of this point of intersection with equations 33 and 34.

In the DVR case we did not need the fact that a_1 was a unit in R, but we use this fact when dealing with a local ring O.

Let $\{r_i\}$ be a the subset of the $\{t_i\}$ such that f has a reduction type I_n over R_i,

Since f is already in pre-chart form, we have for each such r_i, $v_i(a_3) \geq 1$, $v_i(a_4) \geq 1$, and $v_i(a_6) \geq 2$.

In the discussion which follows I will use just two elements r and s of this set, but the argument is the same regardless of the number of elements present in the set. Again since f is already in pre-chart form, we must ensure that all further translations of x and y are multiples of rs. We consider the polynomial

$$y^2 + a_1xy + \frac{a_3}{rs}y = a_2x^2 + \frac{a_4}{rs}x + \frac{a_6}{r^2s^2}. \tag{161}$$

If this polynomial defines a degenerate quadric modulo r or modulo s, we must translate the singular point to $(0, 0)$. For either r or s we may use equations 33 and 34 to find the coordinates of the singular point, and by these equations we remark that these values of x and y can be lifted to an O integral elements. Furthermore if the polynomial is degenerate both modulo r and modulo s, then the coordinates of the singularity must agree modulo (r, s). This means we can find a single element x_0 and a a single element y_0 such that (x_0, y_0) reduces to the double point modulo r and modulo s. We then translate x by $-rsx_0$ and y by $-rsy_0$ Now all further translations of of x and y must be multiples of r^2s^2.

Assuming that the polynomial 161 is degenerate both modulo r and modulo

s, we would next examine

$$y^2 + a_1xy + \frac{a_3}{r^2s^2}y = a_2x^2 + \frac{a_4}{r^2s^2}x + \frac{a_6}{r^4s^4}. \tag{162}$$

By the same argument there would be a single element x_0 such that (x_0, y_0) reduces to the double point modulo r and modulo s. We then translate x by $-r^2s^2x_0$ and y by $-r^2s^2y_0$ Now all further translations of of x and y must be multiples of r^3s^3.

We repeat the process as long as the polynomial is degenerate modulo r or modulo s. As in the previous case, we stop dividing by r or s once the quadric is not singular. The same argument works for any number of parameters r_i and s_i.

11.2.2 a_2 Group

The group contains a variety of different reduction types. It contains members of the K_n family, members of the I_n family, and members of the I_n^* family. For every reduction type in this group we have $v(a_1) \geq 1$. This means in particular that a member of the I_n family in this group must have a residue field of characteristic 0. To simplify the discussion in this section, we may include the reduction types T_n in the I_n^* family, and the reduction types K_n' in the K_n family.

In addition to the fact that every reduction type in this group has $v(a_1) \geq 1$, also know that for the K_n or I_n types we have $v(a_2) = 0$, and for members of the I_n^* family we have $v(a_2) = 1$.

We recall that to translate to chart form in the case of an elliptic scheme over a DVR R with uniformizer π we just needed to translate by the possible singularities of quadratics of the form

$$y^2 + a_1xy + \frac{a_3}{\pi}y = a_2x^2 + \frac{a_4}{\pi}x + \frac{a_6}{\pi^2} \tag{163}$$

in the case of a type I_n, by a rational points of double lines such as

$$y^2 = a_2x^2 + \frac{a_6}{\pi^2} \tag{164}$$

in the case of a type K_n, and by double roots of the quadratic polynomials

$$y^2 + \frac{a_3}{\pi^2}y - \frac{a_6}{\pi^4} \tag{165}$$

$$\frac{a_2}{\pi}x^2 + \frac{a_4}{\pi^3}x + \frac{a_6}{\pi^5} \tag{166}$$

for members of the I_n^* family.

Because $b_2 = a_1^2 + 4a_2$ is a unit, if the quadric is singular it consists of two intersecting lines. In fact we can solve for the coordinates of this point of intersection with equations 33 and 34.

We now perform the analysis when dealing with a local ring O.

Let $\{r_i\}$ be a the subset of the $\{t_i\}$ such that f has a reduction type I_n over R_i, and $\{s_i\}$ be a the subset of the $\{t_i\}$ such that f has a reduction type K_n or K_n' over R_i and $\{p_i\}$ be a the subset of the $\{t_i\}$ such that f has a reduction type I_n^* or T_n over R_i.

Then for each r_i and s_i we have $v_i(a_2) = 0$, $v_i(a_3) \geq 1$, $v_i(a_4) \geq 1$, and $v_i(a_6) \geq 1$ and for each p_i we have $v_i(a_2) = 1$, $v_i(a_3) \geq 2$, $v_i(a_4) \geq 3$, and $v_i(a_6) \geq 4$. In the discussion which follows I will use only one index $r = r_i$, one index $s = s_i$, and one index $p = p_i$ but the argument is the same regardless of the number of indices present in the sets.

Since f is already in pre-chart form, we must ensure that all further translations of x and of y are multiples of rsp^2. There are four types of translations that must be performed in sequence. This sequence is repeated as many times as necessary. In step 1 we consider the polynomial

$$y^2 + a_1xy + \frac{a_3}{rsp^2}y = \frac{a_2}{p}x^2 + \frac{a_4}{rsp^3}x + \frac{a_6}{r^2s^2p^4}. \tag{167}$$

Note that $\frac{a_2}{p}$ is a unit in O. We will use this fact to conclude that a_2 is not a square in the residue fields modulo r or modulo s.

If this polynomial defines a degenerate quadric modulo r we must translate the singular point to $(0, 0)$. We may use equations 46 and 47 to find the coordinates of the singular point, and by these equations we remark that these values of x and y can be lifted to an O integral elements. If there were more than one element r_i in the analysis we could find a a single element x_0 and a a single element y_0 and perform one translation. We then translate x by $-rsp^2x_0$ and y by $-rsp^2y_0$ Now all further translations of of x and y must be multiples of r^2sp^2.

In step 2 we consider the polynomial

$$y^2 = a_2x^2 + \frac{a_6}{r^2s^2p^4} \tag{168}$$

If this defines a double line modulo s with a rational point (x_0, y_0) we need to translate it to $(0, 0)$. Because of the previous step $\frac{a_6}{r^2s^2p^4}$ is zero modulo r

Given such a point (x_0, y_0), we use the fact that a_2 is not a square modulo s to conclude that both x_0 and y_0 can be lifted to O integral values. Furthermore since the constant term of the quadratic is zero modulo r, we have modulo r

$$y_0^2 = a_2x_0^2 \tag{169}$$

Now since a_2 is not a square modulo r, we then also conclude that x_0 and y_0 are zero modulo r. We then translate x by $-rsp^2x_0$ and y by $-rsp^2y_0$. By the last comment these translations were indeed multiples of r^2sp^2. Now all further translations of of x and y must be multiples of $r^2s^2p^2$.

In step 3 we consider the polynomial

$$y^2 + \frac{a_3}{rsp^2}y - \frac{a_6}{r^2s^2p^4} \tag{170}$$

If this polynomial is zero modulo p, we will need to perform a y translation so that this member of the I_n^* family will be in chart form. We notice that both the linear and constant term are zero modulo r and modulo s. Because this polynomial is monic with integral coefficients we see that any such double root can be lifted to an O integral element y_0 which is zero modulo r and modulo s. We then translate y by $-rsp^2y_0$. By the last comment this translation is indeed a multiple of $r^2s^2p^2$. Now all further translations of of y must be a multiple of $r^2s^2p^3$.

In step 4 we consider the polynomial

$$\frac{a_2}{p}x^2 + \frac{a_4}{rsp^3}x\frac{a_6}{r^2s^2p^5} \tag{171}$$

If this polynomial is zero modulo p, we will again need to perform an x translation so that this member of the I_n^* family will be in chart form. We notice that both the linear and constant term are zero modulo r and modulo s. Because this polynomial has leading coefficient $\frac{a_2}{p}$, which is a unit in O, and also has integral coefficients we see that any such double root can be lifted to an O integral element x_0 which is zero modulo r and modulo s.

We then translate x by $-rsp^2x_0$. By the last comment this translation is indeed a multiple of $r^2s^2p^2$. Now all further translations of of x must also be a multiple of $r^2s^2p^3$.

At this point we repeat the four step process as many times as needed. At each stage we remove an extra power of r, s, or p from each of a_3 and a_4, and two such powers from a_6. Of course once the quadratic 167 is non singular modulo r we no longer remove a powers of r. And once the quadratic 168 is not a double line modulo s we no longer remove a powers of s. And once the polynomials 169 and 170 have no double roots modulo p we no longer remove a powers of p. The same argument works for any number of parameters r_i, s_i, and p_i.

For example, supposing we did perform a translation in each of the steps above, we would begin the next stage by examining the polynomial

$$y^2 + a_1xy + \frac{a_3}{r^2s^2p^3}y = \frac{a_2}{p}x^2 + \frac{a_4}{r^2s^2p^4}x + \frac{a_6}{r^4s^4p^6} \tag{172}$$

and check if it defines a degenerate quadratic modulo r.

11.3 Conclusion

Thus we conclude that if all of our reduction types are members of the a_1 group, the $a_1 - a_2$ group , or the a_2 group, we can find a single O translation for x and a single O translation for y, that puts the Weierstrass equation f can be put into multiple chart form. This concludes the sketch of the proof of proposition 11.1.

This strengthens our result of multiple pre-chart form under the pre-settled hypothesis. This will allow us to refine the groups associated to the families of reduction types. By defining the notions of a *wild* and *tame* part of the discriminant and the notion of settled, we will be able to further limit collisions in among the types in the families.

Reducing the number of types of possible collisions puts us in the favorable situation where the type of exceptional divisor of all blow ups of the base is uniquely determined by the reduction types involved in the collision. We will then apply a combinatoric argument in section 14 to further limit ourselves to a small list of collisions. These will eventually be analyzed individually in section 16.

We now turn toward the definition of a settled elliptic scheme and explore the ramifications of such a further hypothesis.

12 A Settled Base

12.1 Tame and Wild Discriminant

To provide a refinement of the notion of a pre-settled scheme and to refine the groups described in section 10 we will introduce the notion of a settled point in the base. Somewhat analogous to the notion of pre-settled, the notion of settled is also one the locally relates the valuation of the discriminant d at each of the DVRs R_i, to that of the valuation of d in the DVR of the exceptional divisor defined by a blow up.

We are going to use the conductor of an elliptic curve as motivation for a definition of the wild part of the discriminant. We recall that in the situation of an elliptic scheme over a DVR of characteristic zero the valuation of the discriminant can be determined just by knowing the reduction type. In fact, if the elliptic curve has additive reduction, Ogg's formula [SIL 2] says $v(d) = 1 + c$ where c is the number of components in the special fiber. If we are in residue characteristic 2 or 3, and have one of the standard Kodaira types, Ogg's formula also says $v(d) = t + w - 1$, where t is the tame part of the conductor, and w is the wild part. Using this terminology as motivation, the definition of tame part of the discriminant will be equivalent to

$$tame(E) = v(d) - w(d). \tag{173}$$

for the standard Kodaira reduction types.

More formally, the tame and wild parts of the discriminant will be defined in an analogous way to the way that the straight and residual parts of the discriminant were defined.

Definition 12.1 (Tame Part of d, Char 2)
Let E be a Weierstrass elliptic scheme defined over a DVR R defined by a Weierstrass equation f in in chart form. Define the <u>*Tame Part of the Discriminant*</u> *$Tame(E)$ to be the minimum of the valuations of the elements*

$$a_6^2 \quad a_4^3 \quad a_3^4 \quad a_2^3 a_6 \quad a_2^2 a_4^2 \quad a_2^3 a_3^2 \quad a_1^6 a_6 \quad a_1^4 a_4^2 \quad a_1^3 a_3^3. \tag{174}$$

If the characteristic of the residue field is not 2, we can replace the list of elements with a shorter one

$$a_6^2 \quad a_4^3 \quad a_2^3 a_6 \quad a_2^2 a_4^2. \tag{175}$$

We make two observations. If the reduction type is not a member of one of the infinite families, the tame part of the discriminant is equal to the straight part of the discriminant. Also if the reduction type is a standard Kodaira type, then the tame part of the discriminant is equal to the entire valuation of the discriminant. That is, for the old reduction types $tame(E) = v(d)$, and the new types have the following values for $tame(E)$.

New Type	*Char*	$tame(E)$
$Z1$	3	0
$Z2$	3	6
$X1$	2	0
$X2$	2	6
$Y1$	2	0
$Y2$	2	4
$Y3$	2	8
K_n *n odd*	2	n
K_n *n even*	2	n
K'_n *n even*	2	n
T_n	2	$6+n$

(176)

We are also going to define a wild part of the discriminant

Definition 12.2 (Wild Part of d)
Let E be a Weierstrass elliptic scheme defined over a DVR R. Then define the Wild Part of the Discriminant by

$$wild(E) = v(d) - tame(E) \tag{177}$$

When we have an elliptic scheme defined over a higher dimensional local ring O, we define these measures for each DVR R_i as follows.

Definition 12.3 (Indexed Tame and Wild Parts)
Let E be a Weierstrass elliptic scheme defined over a regular n dimensional local ring O. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for O. Then for each i we consider E_i over R_i and define $\underline{tame_i(E) = tame(E_i)}$, and $\underline{wild_i(E) = wild(E_i)}$.

Clearly for all i,

$$wild_i(E) = v_i(d) - tame_i(E). \tag{178}$$

As in the section defining the straight and residual parts of the discriminant, we may need to replace our Weierstrass elliptic scheme with a more minimal one. Thus $tame(E)$ is also only defined modulo 12, and we choose for convenience to deal with $wild(E)$, which does not change when passing to a more minimal scheme.

The terminology was chosen because the integer $wild(E)$ corresponds to the wild part of the conductor in case we have on of the standard Kodaira types.

12.2 Settled Fault Integer

We define the notion of pre-settled to obtain the property that if two or more reduction types in the same group collide and we blow up the base at the intersection of the two components, the exceptional divisor has a reduction type in the same group. We will strengthen the notion of a nice collision so that the type of the exceptional divisor will be even more precisely described.

With the definition of the tame and wild part above, these *nice* collisions have the property that after a blow up and before replacing the Weierstrass equation with a more minimal one, the following measures in R_e of the discriminant are related to the same measures in the DVRs R_i as follows.

$$v_e(d) = \sum_{j=1}^{k} v_j(d) \tag{179}$$

$$tame_e(E) = \sum_{j=1}^{k} tame_j(d) \tag{180}$$

$$wild_e(E) = \sum_{j=1}^{k} wild_j(d) \tag{181}$$

We now define a measure of the failure of the relations 180 and 181.

Definition 12.4 (Settled Fault Integer)
Let E be a Weierstrass elliptic scheme defined over a regular n dimensional local ring O. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters for O. Let $B \to Spec(O)$ be the blowing up at the ideal $(t_1, ...t_l)$, and let v_e be the valuation associated to exceptional divisor. Then we define <u>*The Settled Blowup Fault*</u> *at O to be the difference*

$$Fault(E) = wild_e(E) - \sum_{j=1}^{l} wild_j(E) \tag{182}$$

Define <u>$SettledFault_p(E)$</u> *at O_p to be the maximum over all such blowups.*

I remark that the maximum can be reached by blowing up at the closed point of O.

12.3 Defining Settled

A point in the base B is *settled* if the wild part of the discriminant is additive. Equivalently there is no Settled Fault at p.

Definition 12.5 (Settled)
Let $X \to B$ be a Weierstrass elliptic scheme defined over B. Let $p \epsilon B$. We say $X \to B$ is <u>*Settled at p*</u> *if $SettledFault_p(d) = 0$. An elliptic scheme is* <u>*Settled*</u> *if it is settled at every point in the base.*

This notion complements the notion of pre-settled by further limiting the types of collisions within groups involving the infinite families of reduction types. It further forces the appropriate a_i to be $\{t_i\}$ normal elements, and elliminates other spot singularities. Combined with the pre-settled assumption, it makes the reduction types of the exceptional divisor in a blow up completely predictable.

12.4 Use of Rational Maps

In this section we are going to define some rational maps. In the settled case these will turn out to be morphisms. By considering the fibers of the morphism, we will have a more precise and general definition of the groups of reduction types. So for each p in the base Let O_p be the local ring at p, and define a rational map from O_p to $P^2(O_p)$ as follows.

Definition 12.6 (Rational Maps)
Let E be a Weierstrass elliptic scheme defined by a Weierstrass equation f over a regular n dimensional local ring O_p. Suppose that f is in multiple chart form. Define rational maps by

$$\phi_1(p) = (a_1^6 a_6, a_1^4 a_4^2, a_1^3 a_3^3) \tag{183}$$

$$\phi_2(p) = (a_2^3 a_6, a_2^2 a_4^2, a_2^3 a_3^2). \tag{184}$$

Note that ψ is only defined locally and depends on the Weierstrass equation f. We are only going to consider the above rational maps maps at points of the base corresponding to the groups where either a_1 or a_2 is a unit. If fact we can limit the definition of ϕ_1 to points of the base with a_1 a unit, and we can limit the definition of ϕ_2 to points of the base with a_2 a unit. If we assume that our Weierstrass equation f is in chart form, the subschemes $a_1 \neq 0$ and $a_2 \neq 0$ of the base are well defined even though a_1 and a_2 vary with translations.

As with the rational map ψ, we compute the values of the rational maps in the settled case. I modify the argument of section 10.7 to show that ϕ_1 and ϕ_2 are also morphisms.

Let $\{t_i\}$ be a set of discriminant compatable unformizing parameters for O_p. For each index i cancel out the highest power of t_i in the definitions of ϕ_1 and, ϕ_2. For each index i this highest power is equal to the tame part of the discriminant in the DVR R_i.

Supposing, that after this division, one of the components in the definition of ϕ_1 or ϕ_2 is a unit. Then the rational map is well defined at p, and the value $\phi_1(p)$ or $\phi_2(p)$ is just given by the reduction of the reduced components to the residue field of O_p.

Now suppose that after that division every component of ϕ_1 or ϕ_2 is still in the maximal ideal of O_p. Then we consider blowing up the base at the maximal ideal of O_p, and choose any point q on the exceptional divisor. Let t be the uniformizer corresponding to the exceptional divisor. Note that ϕ_1 or ϕ_2 can be defined on the whole blow up of O_p.

When we consider ϕ_1 or ϕ_2 in one of the coordinate patches of the blow up, we see that one power of t can be factored out of the five-tuple defining ϕ_1 or ϕ_2 for each power of t_i that was previously factored out, plus at least one extra power of t.

This is because by equation 133 the valuation of an $x \epsilon O_p$ at t corresponds to the highest power of the maximal ideal that contains x.

In other words, before possibly replacing the Weierstrass equation with a more minimal one,

$$tame(E) > \sum_{j=1}^{k} tame_j(d) \tag{185}$$

Or equivalently

$$wild(E) < \sum_{j=1}^{k} wild_j(d). \tag{186}$$

This violates the settled condition, and is thus a contradition.

This analysis gives us the following

Proposition 12.7 (Rational Maps are Morphisms)
Let E be a Weierstrass elliptic scheme defined over B Suppose the elliptic

scheme is pre-settled and settled. Then ψ is a morphism. At points of B where the first component of ψ is a unit, ϕ_1 is also a morphism. At points of B where the second component of ψ is a unit, ϕ_2 is also a morphism.

So we conclude that the local rational maps are well defined morphisms. Next we will consider fibers of these morphisms to give a better description of what reduction types can collide in the settled case.

12.5 Fibers of the Morphisms

12.5.1 a_1 is a unit

The analysis of this subsection applies only when the local ring O has residue characteristic 2. At points of B where $\psi(p)$ has a unit value in the first component, we consider the morphism ϕ_1 and we could make a chart of which types arise over the various fibers of ϕ_1. I omit this chart, because the only types which do arise regardless of the value of ϕ_1 are the types I_n. The chart would only serve to indicate the valuations of a_3, a_4, and a_6, and state which would be forced to be $\{t_i\}$ normal elements. This would be useful in ultimitely proving regularity at the point of collision.

Even without such an analysis we can see that because the discriminant has normal crossings, and the I_n types have no *wild* part of the discriminant that the map ϕ_1 would be forced to be a morphism even in the absence of the settled hypothesis. Furthermore, if we blow up the base at a collision of I_n types, The exceptional divisor is always of type $I_{n'}$ where n' is just the sum of the n's.

I reserve a more careful analysis of the fibers to the following case where more interesting phenomenon arise.

12.5.2 a_2 is a unit

In this section we assume that $\psi(p)$ has a unit value in the second component, and is zero in the first component. If a_1 and a_2 are are both units we can refer to the previous section. We now make a chart of the fibers of the morphism ϕ_2 which tells us what reduction types can be present over points with a specific value of ϕ_2.

Proposition 12.8 (Settled Collision Families in Char 2)
Let E be a pre-settled and settled elliptic scheme defined by a Weierstrass equation f over a local ring O_p of residue characteristic 2. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O_p. Suppose the second component of ψ is a unit but the first component of ψ is zero. Then if $\phi_2(p)$ is as in the first column of Chart 187, the only possible reduction types T_i over each R_i are as in the second column of Chart 187. Furthermore the the a_i mentioned in the third column must be $\{t_i\}$ normal elements in O_p.

$\phi_2(p)$	*Types*	a_i *normal*
$(*,0,0)$	$I_{2n}, K_{2n}, I^*_{2n+1}$	a_3
$(0,*,0)$	I_{2n}, K_{2n}, I^*_{2n}	a_4
$(0,0,*)$	$I_n, K_{2n+1}, K'_n, T_n, I^*_n$	a_6
$(*,*,0)$	I_{2n}, K_{2n}	a_3, a_4
$(0,*,*)$	I_{2n}, K_{2n}, I^*_{2n}	a_4, a_6
$(*,0,*)$	$I_{2n}, K_{2n}, I^*_{2n+1}$	a_3, a_6
$(*,*,*)$	I_{2n}, K_{2n}	a_3, a_4, a_6

(187)

The I_n types must all have residue characteristic 0, and the I^*_n types in the a_6 group here must also have residue characteristic 0.

12.6 Notes

The results of this section are really only used to describe the collisions in the families of reduction types. If the elliptic scheme had no such reduction types appearing we could drop the settled hypothesis.

Although the pre-settled hypothesis already prevents collision between the groups described in section 10, the settled hypothesis further restricts the types of collisions among reduction types in the infinite families.

Specifically we take a collision of the form in Chart 187 and apply Tate's algorithm to each component of the discriminant locus. The fact that one or more of a_3, a_4, and a_6, are $\{t_i\}$ normal elements guarantees that that in the last blow up of section 16.8, the elements a'_j obtained from the a_j by dividing out by powers of t_i are units. This proves that all points in the total space X lying above the collision point p are regular.

13 Stability of Settled Schemes

13.1 Special Weierstrass Form

We use the pre-settled and settled assumptions via the local morphisms ψ, ϕ_1, and ϕ_2 to severely restrict the form of the Weierstrass equation. In this section we explore the ramifications of such a restricted Weierstrass equation. In particular we define the modulus of a collision and assign integer invariants to the components of the discriminant locus. These integers will be use later in the combinatorial arguments of section 14. We also show that after a blow up of the base, the derived Weierstrass equations are automatically in chart form. Finally, we show that the Weierstrass elliptic scheme obtained by pulling back to the new base is also pre-settled and settled.

We have defined the groups of reduction types in definitions 10.2 and 10.3 and secondly by the fibers of the morphisms ψ. With the pre-settled assumption, proposition 10.13 reduces to the case of collision of reduction types in the same group. Furthermore, for each such group one or more of the a_i are $\{t_i\}$ normal elements. If a_i is a $\{t_i\}$ normal element, its valuation is additive in blow ups. This is useful because with enough information about the valuations of the a_i's we can use the charts to determine what the reduction type the exceptional divisor is.

We start with a settled and pre-settled Weierstrass elliptic scheme and suppose the Weierstrass equations is locally in chart form. After each blow up of the base we pull back the Weierstrass equation to the new base. We will see that the assumptions of settled and pre-settled guarantee that no further translations will be needed to put the new Weierstrass equations in chart form. We may, however need to replace the Weierstrass elliptic scheme with a more minimal one as in section 8.6. We construct this minimal scheme when t^i divides a_i for $i = \{1, 2, 3, 4, 6\}$. For this reason the combinatorial arguments in section 14 consider the addition of the valuations of the a_i modulo one of these integers.

13.2 Mod n Collision Addition

We now consider settled and pre-settled Weierstrass elliptic schemes locally in order to assign integer invariants to the reduction types and to the collisions. Suppose the elliptic scheme is defined at p by a Weierstrass equation f in chart form with respect to a discriminant compatible set of uniformizing parameters $\{t_i\}$. By considering the fibers of the morphisms ψ, we know that the reduction type of each component in the discriminant divisor belongs to the same group. We refer to chart 153 or 154 to determine which a_i are forced to be $\{t_i\}$ normal elements. Suppose m is the greatest common divisor of these indices. We assign the integer $v(a_m)$ to each reduction type in the discriminant locus that passes through p, and will be performing addition modulo m on these components. We now formalize this assignment.

Definition 13.1 (Modulus of Collision)
Let E be a pre-settled and settled elliptic scheme defined by a Weierstrass equation f over a local ring O. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O, and suppose f is in multiple chart form. As specified in Chart 153, Chart 154, some of the a_i are forced to be $\{t_i\}$ normal elements. Define the Modulus of a Collision to be the greatest common divisor of these indices i. Denote this index by m.

As mentioned in relation 144, the vector $(v(a_1), v(a_2), v(a_3), v(a_4), v(a_6))$ is considered modulo the vector $(1, 2, 3, 4, 6)$. On way to interpret $v(a_m)$ is via its simple relationship to the straight part of the discriminant.

Proposition 13.2 (Straight Discriminant)
Let E be a pre-settled elliptic scheme defined by a Weierstrass equation f over a local ring O_p. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O_p, and suppose f is in multiple chart form. Let m be the modulus of a collision at p. Then the straight parts of the discriminant are determined by a_m.

$$straight_i(E) = v_i(a_m)\,\frac{12}{m}. \tag{188}$$

The proof follows from the definition of the straight part of the discriminant and the charts in section 4.

We prefer to phrase our arguments in terms of $v(a_m)$ rather than $straight(E)$, so we now consider how the integer $v(a_m)$ behaves in blow ups.

Proposition 13.3 (Collision Arithmetic)
Let E be a pre-settled elliptic scheme defined by a Weierstrass equation f over a local ring O. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O, and suppose f is in multiple chart form. Let m be the modulus of the collision at p. Let $B \to Spec(O)$ be the blowing up at an ideal $(t_1, ...t_l)$. Let v_e be the exceptional valuation. After reducing the Weierstrass equation to a minimal one we have

$$v_e(a_m) = \sum_{i=1}^{i=l} v_i(a_m) \; mod \; m. \tag{189}$$

To prove that the two integers in 13.3 are equal before reducing modulo m, we just remark that a_m is a $\{t_i\}$ normal element and apply relation 136.

However, we can be more explicit and show $v_e(a_m)\epsilon\{0, ..., m-1\}$ by examining the valuations of the a_i after a blow up, and consider what happens when passing to a minimal scheme. Because a_m is a $\{t_i\}$ normal element, we know the exact form of a_m in O.

$$a_m = \alpha \prod_{k=1}^{k=n} t_k^{e_k}. \tag{190}$$

Here α is a unit in O, and e_k is just the valuation $v_k(a_m)$.

$$e_k = v_k(a_m) = \frac{m}{12} straight_k(E). \tag{191}$$

In fact, before passing to a minimal scheme, all coefficients of the Weierstrass equation satisfy

$$v_e(a_i) \geq \sum_{k=1}^{k=l} v_k(a_i) \geq \frac{i}{12} straight_e(E). \tag{192}$$

This follows from relation 9.6 and the definition of $straight_e(E)$. Thus for all i we have the inequality

$$\frac{v_e(a_m)}{m} \leq \frac{v_e(a_i)}{i}. \tag{193}$$

This means that after we reduce the vector of valuations

$$(v(a_1), v(a_2), v(a_3), v(a_4), v(a_6))$$

modulo the vector $(1, 2, 3, 4, 6)$ as in section 8.6, we have $v(a_m) < m$. This completes the proof of proposition 13.3.

In particular if we are blowing up the base at the intersection of two hypersurfaces H_1 and H_2 defined by the ideal (t_1, t_2) then the valuation of a_m in the exceptional divisor is just

$$v_e(a_m) = v_1(a_m) + v_2(a_m) \quad mod\ m. \tag{194}$$

And if we are blowing up the base at the intersection of three hypersurfaces H_1, H_2 and H_3 defined by the ideal (t_1, t_2, t_3) then the valuation of a_m in the exceptional divisor is just

$$v_e(a_m) = v_1(a_m) + v_2(a_m) + v_3(a_m) \quad mod\ m. \tag{195}$$

We also assign these integer invariants $v(a_m)$ to the components of the discriminant divisor involved in a collision.

For example, consider a local ring O_p and suppose $\psi(p) = (0, 0, 0, 0, 1)$. By chart 154, each reduction type at such a point p must be in the a_6 group, and the modulus of the collision is 6. Then we assign an integer between 0 and 5 to each reduction type in the collision depending on the valuation $v(a_6)$. We formalize this idea with a definition, and in particular make a distinction between types which have $v(a_m) = 0$ and the other types.

Definition 13.4 (Zero and Main Types)
Let E be a pre-settled and settled elliptic scheme defined by a Weierstrass equation f over a regular local ring O_p. Let $\{t_i\}$ be a discriminant compatible

set of uniformizing parameters for O and suppose f is in multiple chart form. Let m be the modulus of the collision at p. Let T_i be the reduction type of the elliptic schemes E_i over R_i. A reduction type T_i is called a Zero Type in the collision if $v_i(a_m) = 0$, and a Main Type in the collision if $v_i(a_m) > 0$. We also call T_i an N Type if $v_i(a_m) = N$.

Notice that although $v_i(a_m)$ is assigned a-priori to a component of the discriminant divisor in an elliptic scheme over a local ring, this integer may be assigned to a whole component of the discriminant divisor in well defined manner. This is because of the well behaved theory of divisors on the base. In other words, the divisor (a_m) on a non-local base has fixed order of vanishing along each component of the discriminant divisor.

13.3 Stability of Chart Form

Before we systematically blow up at the intersection of two and three hypersurfaces, we must guarantee that we can continue to use proposition 13.3 after a blow up. That is, we need the conditions of the proposition to hold for points in the exceptional divisor of a blow up.

We first set up notation to consider blowing up the base at a local ring. Suppose E be an elliptic scheme defined by a Weierstrass equation f over a regular local ring O. Let $B \to O$ be a blow up and O'_p be a local ring in the exceptional divisor. We have already mentioned in proposition 9.5 that we can easily obtain uniformizing parameters for O'_p from the $\{t_i\}$. We now remark that as the Weierstrass equation is pulled back to O'_p it remains in chart form with respect to the new uniformizing parameters.

Proposition 13.5 (Stability of Chart Form)
Let E be a pre-settled elliptic scheme defined by a Weierstrass equation f over a regular local ring O. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O, and suppose f is in multiple chart form. Let $B \to Spec(O)$ be the blowing up at an ideal $(t_1, ...t_l)$. Let O'_p be a local ring of B with a discriminant compatible set of uniformizing parameters u_i obtained as in proposition 9.5. Then the Weierstrass equation f' obtained by pulling

back f to O'_p is in pre-chart form. If we assume also that E is settled elliptic scheme then f' is in full chart form.

To prove this we need only to check that the elliptic scheme over R_e obtained by localizing at the exceptional divisor is in pre-chart or chart form.

I sketch a proof. To show that f' is in pre chart form, we need to check that the pattern of the valuations $v_e(a_i)$ is as for one of the types on a chart in section 4. To understand the pattern of the valuations $v_e(a_i)$ we can use the morphism ψ. We pull back the morphism ψ defined on O to B. Thus ψ is constant on the exceptional divisor of the blow up. We then consider ψ on the generic point on the exceptional divisor to obtain inequalities on the $v_e(a_i)$. These inequalities are of the same form as 192, and force the $v_e(a_i)$ to lie in one of the patterns on chart 4.2, 4.9, or 4.6 depending on the residue characteristic of O. The remaining polynomial conditions, and conditions stating that an an element is not a perfect square or cube can be checked case by case.

To show that f' is in full chart form, we use the morphism ϕ_1 or ϕ_2 in the same way. It also pulls back to B and forces the $v_e(a_i)$ to fall in a pattern of one of the sub charts 4.3, 4.4, 4.5, 4.7, or 4.8. Again the remaining conditions can be checked case by case.

Another way to intuitively understand proposition 13.5 is to assume that f' requires a translation in order to be in pre-chart form. Such a translation shows that

$$v_e(a_m) = \sum_{i=1}^{n} v_i(a_m). \tag{196}$$

Because the $v_i(a_m)$ are linearly related to the straight part of the discriminant as in proposition 13.2, $Straight_e(E)$ is too large, and violates the pre-settled assumption. Similarly, if f is in chart form, other coefficients are also forced to be $\{t_i\}$ normal as specified in section 12.5. We can relate the tame part of the discriminant to these valuations. A potential translation in this case would cause $Tame_e(E)$ to be too large, and thus violate the settled assumption.

Not only will we be able to repeatedly use proposition 13.3 after a blow up, but proposition 13.6 also shows us that the reduction type of the exceptional divisor is almost completely determined.

Corollary 13.6 (Exceptional Type Determined)
Let E be a pre-settled and settled Weierstrass elliptic scheme defined over a regular local ring O. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O. Suppose each E_i has reduction type T_i. Then the reduction type of the exceptional divisor T_e is determined by the reduction types T_i up to the ambiguity described in the list 197.

Proof. Because E is pre-settled, we already know that the reduction type of the exceptional divisor is in the same group as the type each component in the collision. Also, because E is pre-settled, we can describe it with a Weierstrass equation in chart form, and use relation 192 to calculate equalities or inequalities on the $v_e(a_i)$. In particular, if m is the modulus of the collision, we know the exact valuation $v(a_m)$. This determines the exceptional divisor up to ambiguity in the following sets of reduction types:

$$\begin{gathered} \{I_n, K_n, K'n\} \\ \{I_n^*, T_n\} \\ \{I_0, X1, Y1, Z1\} \\ \{IV, Y2\} \\ \{I_0^*, Z2, X2\} \\ \{IV^*, Y3\}. \end{gathered} \tag{197}$$

This theorem can be significantly strengthened because we can usually determine the exact type by considering the residual or wild part of the discriminant, or by using the fibers of the morphisms ϕ, ψ_1, or ψ_2.

We need one more proposition before we can apply proposition 13.3 to a series of blow ups and calculate the valuation $v_e(a_m)$, and reduction type of the exceptional divisor with corollary 13.6. We need to know that our elliptic scheme remains pre-settled and settled after each blow up.

Proposition 13.7 (Stability of Settled)
Let E be a pre-settled elliptic scheme defined by a Weierstrass equation f

over a regular local ring O. Let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters for O, and suppose f is in multiple chart form. Let $B \to Spec(O)$ be the blowing up at an ideal $(t_1, ...t_l)$. Let E' be the minimal Weierstrass elliptic scheme obtained by pulling back E to B. Then E' is a pre settled elliptic scheme. If we assume also that E is settled elliptic scheme then E is also a settled elliptic scheme.

Sketch of Proof: Let O'_p be a local ring of B with a discriminant compatible set of uniformizing parameters u_i obtained as in proposition 9.5. Then by 13.5 the Weierstrass equation f' obtained by pulling back f to O'_p is in multiple pre-chart form (respectively multiple chart form). We also know that a_m is a u_i normal element and if f is in multiple chart form some of the other a_j's are also u_i normal elements. Consider any further blow up at O'_p. In any local ring of this second exceptional divisor we can create a new Weierstrass equation f'' which is automatically by proposition 13.5 in multiple chart form. Because a_m is a u_i normal element we can compute the exceptional straight and tame parts of the discriminant just by looking at the chart. We then conclude that $Fault(p') = 0$ (respectively $SettledFault(p') = 0$). This is true for any second blow up of O'_p, so E' is indeed a pre-settled, (respectively settled) Weierstrass elliptic scheme.

13.4 Summary of Settled Elliptic Schemes

We summarize the nice properties that pre-settled and settled Weierstrass elliptic schemes have. First, there are only collisions between reduction types in the same group. The Weierstrass equations defining them remain in chart form, after being pulled back via a blow up. The minimal Weierstrass elliptic schemes derived from the blow up are themselves pre-settled and settled elliptic schemes. Also the reduction type of the exceptional divisor is almost uniquely determined. Lastly, and important for the following sections, we can assign each component of the discriminant divisor an integer $v(a_m)$ that is additive in blow ups.

We are now ready to algorithmically blow up the base of our elliptic scheme. After the combinatorial arguments involving $v_e(a_m)$, we will make charts

198 and 200 of all remaining collisions that need to be considered in the desingularization of the total space.

14 Combinatorial Reductions

14.1 Finiteness Considerations

In this subsection we provide an algorithm to further limit collisions of reduction types. An elliptic scheme constructed with this algorithm will be called a Limited Weierstrass Elliptic Scheme. In section 16, we will be able to desingularize a Limited Weierstrass Elliptic Scheme.

All blowups will be along subschemes defined or two of three components of the discriminant locus. This defines a sheaf of ideals on B for the blow up. Locally, these blow ups will be at ideals such as (t_1, t_2) or (t_1, t_2, t_3) where the $t_i = 0$ are components of the discriminant divisor at a point $p \epsilon B$.

We will simply describe the blow ups by indicating at the intersection of which hypersurface H_i we are blowing up at, and we will analyze the results locally. These algorithms will take place in stages: one stage for each possible modulus of the collisions. For each integer $m \epsilon \{1, 2, 3, 4, 6\}$ we define a subset of the the components of the discriminant locus.

Definition 14.1 *Let $\underline{S(m)}$ to be the set of hypersurfaces H in the discriminant locus such there exists a point $p \epsilon H$ with modulus$(p) = m$. Call this subset of components $\underline{\text{The Net of Modulus } m \text{ Components}}$.*

Because B is Noetherian, there are only a finite number of components of the discriminant divisor, and thus $S(m)$ is also finite.

To be concrete, perform the algorithms in section 14.2 for the nets of modulus m components in the following order: $S(6)$, $S(4)$, $S(3)$, $S(2)$, $S(1)$. This order could a-priori matter because two meeting components of the discriminant locus could be in more than one net $S(m)$.

14.2 Combinatorial Arguments

14.2.1 Modulus 1

In this section we consider blow ups defined by hypersurfaces H in the net of modulus 1 component, $S(1)$. Each such hypersurface contains a point $p\epsilon H$ of collision modulus 1.

Referring to chart 154, we see that the only possible reduction types are the I_n types. Although all types are trivially zero types, we still perform some blow ups.

Suppose there is a collision of reduction types I_n and I_m with both n and m odd. Blow up at the intersection of these two hypersurfaces on the base. Then the exceptional divisor is of type I_{n+m}. Thus we have eliminated a collision between two odd I_n types, and we have created no new such collisions. Repeat this blow up for each such pair of reduction types. There are only a finite number of components in the entire discriminant locus, only a finite number of blow ups are required for this operation.

We have reduced the modulus 1 collisions to collisions between I_n types with at most one odd n.

14.2.2 Modulus 2

In this section we consider blow ups defined by hypersurfaces H in the net of modulus 2 component, $S(2)$. Each such hypersurface contains a point $p\epsilon H$ of collision modulus 2.

Referring to charts 153 and 154, we see that the only possible zero types are I_0, I_n, K_n, $X1$, and $Y1$. The only possible 1 types are I_0^*,I_n^*, $X2$, and T_n.

Step 1: Blow up at the intersection of any two 1 types. Then in a collision at most one 1 type can exist.

Step 2: Suppose there is an intersection between a type I_n or K_n, and a second I_m or K_m with both m and n odd.

Then all collisions involve at most one 1 type, and among collisions of only zero types, at most one odd indexed I_n or K_n type appears.

14.2.3 Modulus 3

In this section we consider blow ups defined by hypersurfaces H in the net of modulus 3 component, $S(3)$. Each such hypersurface contains a point $p \epsilon H$ of collision modulus 3.

Referring to charts 154, we see that the only possible zero type is I_0. The only possible 1 type is IV. The only possible 2 type is IV^*.

Step 1: Blow up at the intersection of any two 1-types. This eliminates all $(1,1)$ intersections, creating new $(1,2)$ intersections, and possible $(2,2)$ intersections if the $(1,1)$ intersection specializes to a $(1,1,2)$ triple intersection for some $p \epsilon B$.

Step 2: Blow up at any $(1,2)$ intersections. The exceptional divisor is a zero type, so this eliminates all collisions involving a 1 type. The only collisions that remain are collisions among some number of 2 types.

Step 3: Blow up at any triple $(2,2,2)$ intersections. The exceptional divisor is a zero type, so no new collisions are created. There remain only double $(2,2)$ intersections, and no triple intersections.

Step 4: Blow up at the intersection of any two 2-types. This eliminates all $(2,2)$ intersections, but creates only new $(1,2)$ intersections.

Step 5: Blow up at any of the new $(1,2)$ intersections. The exceptional divisor is a zero type, so this eliminates all collisions involving a 2 type.

Then there are no collisions which involve more than 1 main type, and the only zero type is I_0. Effectively all collisions have been eliminated.

An alternative algorithm blows ups sequentially at the following types of intersections: $(1,2)$, $(2,2,2)$, $(1,1,1)$, $(1,1)$, $(2,2)$, $(1,2)$. We omit further details.

14.2.4 Modulus 4

In this section we consider blow ups defined by hypersurfaces H in the net of modulus 4 component, $S(4)$. Each such hypersurface contains a point $p \epsilon H$ of collision modulus 4.

Referring to charts 153 and 154, we see that the only possible zero types are I_0, $X1$, and $Z1$. The only possible 1 type is III. The only possible 2 types are I_0^*, $X2$ and $Z2$. The only possible 3 type is III^*.

We ignore the zero types that may be present in each collision, but we do consider them in section 14.4.

Step 1: Eliminate $(3,3)$ intersections. New $(2,3)$ intersections are created, Possible new $(1,2)$ and $(2,2)$ intersections are created.

Step 2: Eliminate $(2,3)$ intersections. New $(1,3)$ and $(1,2)$ intersections are created, Possible new $(1,1)$ intersections are created.

Step 3: Eliminate $(1,3)$ intersections. No new intersections of main types are created. Now a 3 type can only collide with zero types.

Step 4: Eliminate $(1,1)$ intersections. New $(1,2)$ intersections are created. Possible new $(2,2)$ intersections are created.

Step 5: Eliminate $(2,2)$ intersections. No new intersections of main types are created.

Now a collision can contain at most two main types, and any number of zero types. If there are two main types in a collision one is a 1 type and the other is a 2 type. There also remain collisions between only one main type and any number of zero types.

14.2.5 Modulus 6

In this section we consider blow ups defined by hypersurfaces H in the net of modulus 6 component, $S(6)$. Each such hypersurface contains a point $p \epsilon H$ of collision modulus 6.

Referring to charts 153 and 154, we see that the only possible zero types are I_0,$X1$,$Z1$, and $Y1$. The only possible 1 type is II. The only possible 2 types are IV and $Y2$. The only possible 3 types are I_0^*, $X2$ and $Z2$. The only possible 4 types are IV^* and $Y3$. The only possible 5 type is II^*.

We ignore the zero types that may be present in each collision, but we do consider them in section 14.4.

Step 1: Eliminate $(5,5)$ intersections. New $(5,4)$intersections are created. Possible new $(4,4)$, $(4,3)$, $(4,2)$ and $(4,1)$ intersections are created.

Step 2: Eliminate $(5,4)$ intersections. New $(5,3)$ and $(4,3)$ intersections are created. Possible new $(3,3)$, $(3,2)$ and $(3,1)$ intersections are created.

Step 3: Eliminate $(5,3)$ intersections. New $(5,2)$ and $(3,2)$ intersections are created. Possible new $(2,2)$ and $(2,1)$ intersections are created.

Step 4: Eliminate $(5,2)$ intersections. New $(5,1)$ and $(2,1)$ intersections are created. Possible new $(2,1)$ intersections are created.

Step 5: Eliminate $(5,1)$ intersections. No new intersections are created.

Now 5 types are not involved in any collisions with other main types.

Step 6: Eliminate $(4,4)$ intersections. New $(4,2)$ intersections are created. Possible new $(3,2)$, $(2,2)$ and $(2,1)$ intersections are created.

Step 7: Eliminate $(4,3)$ intersections. New $(4,1)$ and $(3,1)$ intersections are created. Possible new $(2,1)$ and $(1,1)$ intersections are created.

Step 8: Eliminate $(4,2)$ intersections. No new intersections are created.

Now the only double collisions of main types involving a 4 type are a $(4,1)$ collision.

Step 9: Eliminate $(4,1,1)$ intersections. The exceptional divisor is a zero type so no new intersections are created.

Now there are also no 4 types involved in a triple collision.

Step 10: Eliminate $(3,3)$ intersections. No new intersections are created.

Step 11: Eliminate $(1, 1)$ intersections. New $(2, 1)$ intersections are created. Possible new $(3, 2)$ and $(2, 2)$ intersections are created.

Step 12: Eliminate $(3, 2, 1)$ intersections. The exceptional divisor is a zero type so no new intersections are created.

Step 13: Eliminate $(2, 2, 2)$ intersections. The exceptional divisor is a zero type so no new intersections are created.

If a 3 type is in a double collision it can only be in a $(3, 1)$ or $(3, 2)$ intersection. If a 3 type is in a triple collision it must be a $(3, 2, 2)$ intersection. Also a 3 type is in no quadruple collisions.

Step 14: Eliminate $(3, 2, 2)$ intersections. This creates only new $(3, 2, 1)$ and $(2, 2, 1)$ intersections.

Step 15: Eliminate the new $(3, 2, 1)$ intersections. The exceptional divisor is a zero type so no new intersections are created.

Now the only collisions of main types involving a 3 type are the $(3, 1)$ and $(3, 2)$ collisions. Furthermore, we have no triple $(2, 2, 2)$ intersections, and no $(1, 1)$ intersections.

Step 16: Eliminate $(2, 2)$ intersections. New $(4, 2)$ intersections are created. Possible $(4, 1)$ and $(4, 2, 1)$ intersections are created, but no other types of intersections involving a 4 type.

Step 17: Eliminate the new $(4, 2)$ intersections. No new intersections are created.

There are now no triple intersections at all.

Step 18: Eliminate the $(2, 1)$ intersections. Only new $(3, 2)$ and $(3, 1)$ intersections are created. This step could be considered optional because $(2, 1)$ collisions are easy to deal with.

Thus there are no triple collisions, and only double collisions of type $(4, 1)$, $(3, 2)$, or $(3, 1)$ remain. There may also be collisions between one main type and any additional number of zero types may be present.

14.3 Independence of Reductions

We now point out that the blowups involved for one $S(m)$ do not interfere in the blow ups for a following $S(m)$.

Proposition 14.2 (Independence of reductions)
Suppose the blow ups described above are performed for the $S(6)$, $S(4)$, $S(3)$, $S(2)$, and $S(1)$ sets of components of the hypersurfaces, in that order. Then the blow ups required for a given set $S(m)$ never involve two meeting hypersurfaces in $S(m')$ with $m' > m$.

To prove this we consider the reduction types that are in more than one set $S(m)$.

The only reduction types that are in the $S(6)$ and $S(4)$ sets are types I_0^*, $X2$ or $Z2$. After the $S(6)$ reductions no such two meet. Thus the reductions for $S(4)$ can never blow up at the intersection of two components in $S(6)$.

After the above reductions, no two IV or IV^* types in the $S(6)$ set meet. Thus when performing the $S(3)$ reductions, no blow up involves two meeting IV or IV^* types in the $S(6)$ set.

The only type in the $S(2)$ set that can be in a set $S(3)$, $S(4)$, or $S(6)$ is type I_0^*. But after the above reductions, no such two I_0^* types meet. Thus when performing the $S(2)$ reductions, no blow up involves two meeting I_0^* types if both are in the $S(6)$ or $S(4)$ set.

Finally after the $S(2)$ reductions no two I_n types in $S(2)$ collide with both n's odd. Thus when performing the $S(1)$ reductions, no blow up involves two meeting odd I_n types if both are in the $S(2)$ set.

14.4 Remaining Collisions

We summarize here the results of the mod m arithmetic arguments that limit collisions with two propositions and charts. If we assume that all reductions in the previous section have been made, we have a very special elliptic scheme. We give a name to such Weierstrass elliptic schemes.

Definition 14.3 (Limited Weierstrass Elliptic Scheme)
Let E be a pre-settled and settled Weierstrass elliptic scheme defined over a base scheme B. Let $B' \to B$ be the blow ups described in section 14.2. Let $E' \to B'$ be the minimal Weierstrass elliptic scheme defined by pullback and the minimal scheme construction in section 8.6. Then $E' \to B'$ is called a Limited Minimal Weierstrass Elliptic Scheme.

14.4.1 Characteristic $\neq 2$

Here we describe the remaining collisions at points p of B not of residue characteristic 2.

Proposition 14.4 (Double Collisions in Char $\neq$ 2)
Let E be a Limited Minimal Weierstrass Elliptic Scheme defined over a base scheme B. Let p be a point of B of residue characteristic $\neq 2$.

Let t_i be a discriminant compatible set of uniformizing parameters at p, so that E is cut out by a Weierstrass equation f in chart form, and the morphism ψ is defined at p. Suppose there is a collision at p involving more than one main type. Then the types involved in such a collision are described in chart 198, where $\psi(p)$ must be as in the first column of the chart.

ψ	$t_i\,Normal$	Mod	$Collision$	$Main\,Types$	$Zero\,Types$	
$(0,1,0)$	a_4	4	$(1,2)$	(III, I_0^*)		
$(0,0,1)$	a_6	6	$(1,3)$	(II, I_0^*)	$Z1$	
$(0,0,1)$	a_6	6	$(1,3)$	$(II, Z2)$	$Z1$	(198)
$(0,0,1)$	a_6	6	$(1,4)$	$(II, IV*)$	$Z1$	
$(0,0,1)$	a_6	6	$(2,3)$	(IV, I_0^*)	$Z1$	
$(0,0,1)$	a_6	6	$(2,3)$	$(IV, Z2)$	$Z1$	

We explain the chart further. The first column list the possible values of $\psi(p)$. If the value of $\psi(p)$ is not in this column, no collision involving more than one type occurs at p. The second column indicates which a_j is forced to be a $\{t_i\}$ normal element. The third column gives the modulus m of the collision. The fourth columns indicates $v(a_m)$ for the two types in the collision. Fifth column states what reduction types these $v(a_m)$ types can correspond to, and the last column indicates the type of possible further zero types in the collision.

All other collisions not covered by proposition 198 include at most one main type.

Proposition 14.5 (Single Collisions in Char $\neq 2$)
Let E be a Limited Minimal Weierstrass Elliptic Scheme defined over a base scheme B. Let p be a point of B of residue characteristic $\neq 2$. Let $\{t_i\}$ be discriminant compatible set of uniformizing parameters at p, so that E is cut out by a Weierstrass equation f in chart form, and the morphism ψ is defined at p. Suppose there is a collision at p involving at most one main type. Then $\psi(p) = (1, 0, 0)$ or $\psi(p) = (0, 0, 1)$.

If $\psi(p) = (1, 0, 0)$, the collision is between at most one I_n^* type and one or more I_n types. In this case the modulus of the collision is 2, and a_2 is a $\{t_i\}$ normal element.

If $\psi(p) = (0, 0, 1)$, the collision is between at most one one main type II, IV, I_0^*, $Z2$, IV^*, or II^*, and one or more zero types $Z1$. In this case the modulus of the collision is 6, and a_6 is a $\{t_i\}$ normal element.

For an example, consider the third line of the chart 198. This corresponds to a point p in the discriminant divisor with $\psi(p) = (0, 0, 1)$. The modulus of the collision at p is 6. This entry corresponds to a collision between one 1 type and one 3 type, plus some number or zero types. Concretely, let O be a local ring of characteristic 3 with uniformizing parameters s, t, u such that $t \mid 3$ and $u \mid 3$. The scheme given by

$$y^2 = x^3 + st^3 \tag{199}$$

corresponds to a collision of types II, $Z2$, and $Z1$. These are 1, 3, and zero types respectively.

14.4.2 Characteristic 2

Here we describe the remaining collisions at points p of B of residue characteristic 2.

Proposition 14.6 (Double Collisions in Char 2)
Let E be a Limited Minimal Weierstrass Elliptic Scheme defined over a base scheme B. Let p be a point of B of residue characteristic 2. Let $\{t_i\}$ be discriminant compatible set of uniformizing parameters at p, so that E is cut out by a Weierstrass equation f in chart form, and the morphism ψ is defined at p. Suppose there is a collision at p involving more than one main type. Then the types involved in such a collision are described in chart 200, where $\psi(p)$ must be as in the first column of the chart.

ψ	$t_i\,Normal$	Mod	$Collision$	$Main\,Types$	$Zero\,Types$
$(0,0,0,1,0)$	a_4	4	$(1,2)$	(III, I_0^*)	$X1$
$(0,0,0,1,0)$	a_4	4	$(1,2)$	$(III, X2)$	$X1$
$(0,0,0,0,1)$	a_6	6	$(1,3)$	(II, I_0^*)	$X1, Y1$
$(0,0,0,0,1)$	a_6	6	$(1,3)$	$(II, X2)$	$X1, Y1$
$(0,0,0,0,1)$	a_6	6	$(1,4)$	$(II, IV*)$	$X1, Y1$
$(0,0,0,0,1)$	a_6	6	$(1,4)$	$(II, Y3)$	$X1, Y1$
$(0,0,0,0,1)$	a_6	6	$(2,3)$	(IV, I_0^*)	$X1, Y1$
$(0,0,0,0,1)$	a_6	6	$(2,3)$	$(IV, X2)$	$X1, Y1$

(200)

The chart has the same structure as chart 198. The first column list the possible values of $\psi(p)$. If the value of $\psi(p)$ is not in this column, no collision involving more than one type occurs at p. The second column indicates which a_j is forced to be a $\{t_i\}$ normal element. The third column gives the modulus m of the collision. The fourth columns indicates $v(a_m)$ for the two types in the collision. Fifth column states what reduction types these $v(a_m)$ types can correspond to, and the last column indicates the type of possible further zero types in the collision.

All other collisions not covered by proposition 198 include at most one main type.

Proposition 14.7 (Single Collisions in Char 2)
Let E be a Limited Minimal Weierstrass Elliptic Scheme defined over a base scheme B. Let p be a point of B of residue characteristic 2. Let $\{t_i\}$ be discriminant compatible set of uniformizing parameters at p, so that E is cut out by a Weierstrass equation f in chart form, and the morphism ψ is defined at p. Suppose there is a collision at p involving at most one main type. Then

If there is exactly one main type, $\psi(p)$ must be as in chart 201. The collision is between one main type in the fourth column of chart 201, and one or more zero types in the fifth column. The modulus of the collision and which coefficients are forced to be $\{t_i\}$ normal elements are also specified in the chart.

If there are only zero types, $\psi(p)$ must be as in chart 202. The collision is between two or more zero types in the fourth column of chart 202. The modulus of the collision and which coefficient is forced to be $\{t_i\}$ normal elements are also specified in the chart.

ψ	$t_i\,Normal$	Mod	$Main\,Types$	$Zero\,Types$	
$(0,1,0,0,0)$	a_2	2	I_n^*, T_n	I_n, K_n, K_n'	
$(0,0,0,1,0)$	a_4	4	$III, I_0^*, X2, III^*$	$X1$	
$(0,*,0,*,0)$	a_2, a_4	2	$I_0^*, X2$	$X1$	
$(0,0,0,0,1)$	a_6	6	$II, IV, X2, Y2$ $I_0^*, IV^*, Y3, II^*$	$X1, Y1$	(201)
$(0,0,0,*,*)$	a_4, a_6	2	$I_0^*, X2$	$X1$	
$(0,*,0,0,*)$	a_2, a_6	2	I_0^*	$X1$	
$(0,*,0,*,*)$	a_2, a_4, a_6	2	$I_0^*, X2$	$X1$	

The chart has the similar structure as chart 200, except that there is only one main type in the collision.

ψ	$t_i\,Normal$	Mod	$Zero\,Types$	
$(1,0,0,0,0)$	$a1$	1	I_n	
$(0,1,0,0,0)$	$a2$	2	I_n, K_n, K_n'	(202)
$(0,0,0,0,1)$	$a6$	6	$X1, Y1$	

14.4.3 Double Collisions

If we ignore zero types, the analysis of section 14.2 limits us to the study of double collisions, regardless of the dimension of B'. It is true, however, that any number of zero types can collide at a point, up to the dimension of B. These collisions have been discussed, and are non-trivial in the case of collisions of types in the infinite families, such at types I_n. It is, however, interesting that the limit of the number of main types in a collision is independent of the dimension of the base.

Corollary 14.8 (Only Double Collisions)
Let E be a Limited Minimal Weierstrass Elliptic Scheme defined over a base scheme B. Let p be a point of the discriminant divisor. Suppose that the discriminant has n components at p, each with reduction type T_i. At most two of the T_i can be main types.

The above charts and discussion of zero types tell us what collisions are left to consider. Specifically each line in chart 198 or 200 specifies the form of a collision involving two main types and some zero types. We deal with these types of collisions explicitly in section 16.6.

There are also collisions which involve only one main type. These collisions are covered in section 16.7, unless they involve a type in one of the infinite families. The zero types in these collision are the $X1$, $Y1$, and $Z1$ types, and do not require any blow ups, but the special fiber at the collision point may differ from the generic special fiber of the main type.

The collisions which do involve the infinite families and have at most one main type are discussed in section 16.8. Both the main and zero types in such a collision require one or more blow ups. These zero types are types I_n and K_n, and K'_n.

14.5 Settled Implies Resolvable

An alternative definition for a limited Weierstrass elliptic scheme would be a Weierstrass elliptic scheme with only the double type of collisions in charts 198 and 200, and other the collisions described in sections 14.4.1, and 14.4.2. From this point of view the algorithm above reduce the study of arbitrary Weierstrass elliptic schemes to the study of limited Weierstrass elliptic schemes.

That is, we compose the blow ups this section into one morphism $B' \to B$, pull back the elliptic scheme to B', and consider the birationally equivalent minimal Weierstrass elliptic scheme. We have proved the following theorem.

Theorem 14.9 (Reduction to Limited Elliptic Schemes)
Let B be a regular Noetherian n-dimensional integral separated scheme, Let $X \to B$ be an elliptic subscheme of $P^2(B)$ defined locally by Weierstrass Equations. Suppose X is pre-settled and settled over B. Then there exists a blow up $B' \to B$ defining the base change

$$\begin{array}{ccccc} X_{min} & \leftrightarrow & X' & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ B' & = & B' & \to & B \end{array}$$

and a minimal Weierstrass elliptic scheme X_{min} birational to X' over B' such that X_{min} is a limited Weierstrass elliptic scheme over B'.

Now the minimal Weierstrass elliptic scheme X_{min} over B' is prepared for desingularization. Given such a further blow up $X'' \to X_{min}$, we will have proven theorem 1.2. To produce a regular model X'' in the case of a limited Weierstrass elliptic scheme all we need to do is to perform Tate's algorithm for each component of the discriminant locus, and then check regularity at all points of X'' lying in a fiber above the discriminant locus.

As a result of explicit blow up computations, we can also describe the special fibers of the collisions.

15 The J morphism

15.1 Definition

In this section we specialize to Weierstrass elliptic schemes over bases that have no points of residue characteristic 2 or 3. We make the assumption that there is a morphism on the base that specializes to the j invariant for elliptic curves [SIL 1] for fibers of the elliptic scheme that are non-singular elliptic curves. This technique has also been used by Miranda in [MIR], where he considered elliptic threefolds over surface defined over a field of characteristic zero.

Much of the machinery of this paper has been designed to deal with points of the base that do have points of residue characteristic 2 or 3, so this section also highlights how the analysis simplifies if $1/6$ is in all local rings of the base. For example, we can always work with the simpler Weierstrass equation 1, and the charts of section 4 become much simpler. In particular, none of the new types described in section 6 arise as special fibers. This is because each time a polynomial in Tate's algorithm has a double root over an extension field of κ, the root is actually κ rational. Also, the wild part of the discriminant defined in section 12 is always zero, so the elliptic scheme is trivially settled.

If we can show that the elliptic scheme is also pre-settled, we will be able to directly apply the main theorem 1.2 to show the existence of a regular model. Alternatively, we can prove the result directly by showing that the groups of reduction types as defined in section 10 are separated and that we can perform the reductions in section 14 to show that we have a limited Weierstrass elliptic scheme. Then we would construct the regular model explicitly as in section 16.

Given an arbitrary Weierstrass elliptic scheme we can always consider the open subscheme that is the complement of the divisor (6), and ignore phenomenon that occur over the primes 2 and 3.

We motivate the definition of the J morphism from the j invariant of an elliptic curve. This invariant assigns to any elliptic curve the number c_4^3/d.

The standard formulas for c_4, c_6, and d are summarized in [SIL 1], and we note that c_4, c_6, d and j are translation invariant and thus independent of choice of Weierstrass equation.

We now define a J morphism.

Definition 15.1 (J morphism) *Let B be a regular Noetherian n - dimensional integral separated scheme. Suppose that none of the residue fields of B have residue characteristic 2 or 3. Let $X \rightarrow B$ be an elliptic subscheme of $P^2(B)$ defined locally by Weierstrass Equations. Suppose the discriminant divisor has normal crossings. Define the J map locally to be the rational map $J : B \rightarrow P^1(B)$ given by*

$$p \mapsto (c_4^3, d) \tag{203}$$

If J extends to a morphism $J : B \rightarrow P^1(B)$ then it is called a J morphism.

Note that although the J map is defined by local Weierstrass equations, it is at least a well defined rational map, since c_4, and d are well defined functions on B.

In the following section we assume that the J map is indeed a morphism and we compute J on points of B which are in the discriminant locus. We then show that only collisions among reduction types in the same group occur. We show that this implies that our Weierstrass elliptic scheme is pre-settled. This allows us to apply the main theorem 1.2 to such Weierstrass elliptic schemes.

15.2 Calculating J

In this section we assume that X is a Weierstrass elliptic scheme over a base B, that has no points of residue characteristic 2 or 3, that the discriminant divisor has normal crossings and that there is a J morphism defined on B. We first consider the base to be a regular local ring O.

Because 1/3 is in each local ring, we may translate and use a simpler Weierstrass equation.

$$y^2 = x^3 + a_4 x + a_6. \tag{204}$$

Note that this Weierstrass equation may not be in chart form, as we defined in in section 4.

However, we can make a similar chart for these Weierstrass equations. This chart also appear as an exercise in [SIL 2].

Type	I_0	I_n	II	III	IV	I_0^*	I_n^*	IV^*	III^*	II^*	o/w
$v(a_4)$		0	1+	1	2+	2+	2	3+	3	4+	4+
$v(a_6)$		0	1	2+	2	3+	3	4	5+	5	6+
$v(d)$	0	n	2	3	4	6	$6+n$	8	9	10	12+

(205)

With the simpler Weierstrass equation 204, we can use the definitions of c_4 and d to compute J in terms of a_4 and a_6. We use the relations $c_4 = 48a_4$ and $d = 64a_4^3 - 27 \times 16a_6^2$ to conclude

$$\frac{c_4}{d} = \frac{4 \times 12^3 a_4^3}{4a_4^3 - 27a_6^2}. \tag{206}$$

When $J(p) = (a, b)$, we also write $J = \frac{a}{b}$.

Consider H a component of the discriminant divisor, and let p be a generic point of H that passes through no other components of the discriminant locus. Suppose $\{t_i\}$ is a discriminant compatible set of uniformizing parameters at p, and that $t_1 = 0$ defines the reduced discriminant locus at p.

Because only one component of the discriminant locus passes through p, we know

$$d = \alpha t_1^n. \tag{207}$$

where α is a unit in O_p. Similarly since we chose p to be a generic point on H, we may assume that a_4 and a_6 have the following simple form.

$$a_4 = \beta t_1^n. \tag{208}$$

$$a_6 = \gamma t_1^n. \tag{209}$$

Here β and γ are units in O_p. Note we do not claim that we can do this for all points $p \epsilon H$, but at least for points on some open subset of $U \subset H$.

Then for all types except I_0 and I_0^* we can compute the value of the J morphism just by examining the valuations $v_1(a_4)$ and $v_1(d)$ in chart 205.

For types except I_0 and I_0^* the value of J is constant on an open subset $U \subset H$. Therefore, J is also constant on all of H. We summarize the results of these computations in a proposition.

Proposition 15.2 *Let B be a regular Noetherian n - dimensional integral separated scheme. Suppose that none of the residue fields of B have residue characteristic 2 or 3. Let $X \to B$ be an elliptic subscheme of $P^2(B)$ defined locally by Weierstrass Equations. Suppose the discriminant divisor has normal crossings, and that there is a J morphism $J : B \to P^1(B)$. Suppose p is a point on a component H of the discriminant divisor and that H does not have reduction type I_0^*. Then the valuations $v(a_4)$ and $v(d)$ and the value of J only depend on the reduction type of H and are specified in chart 210.*

Type	$v(a_4)$	$v(d)$	J
I_n	0	n	∞
II	≥ 1	2	0
III	1	3	1728
IV	≥ 2	2	0
I_n^*	1	$6+n$	∞
IV^*	≥ 3	2	0
III^*	3	3	1728
II^*	≥ 4	2	0

(210)

It is not true that J is constant on a component of the discriminant divisor of type I_0^*. By considering the fibers of the J morphism, we observe that collisions among reduction types is limited. For each of the cases $J = 0$, $J = 1728$, $J = \infty$, $J \neq 0, 1728, \infty$, chart 210 shows us that the reduction types must belong in the same group. Thus we have the following analog of proposition 10.13.

Proposition 15.3 (J Morphism Separates Groups)
Let B be a regular Noetherian n - dimensional integral separated scheme. Suppose that none of the residue fields of B have residue characteristic 2 or 3. Let $X \to B$ be an elliptic subscheme of $P^2(B)$ defined locally by Weierstrass Equations. Suppose the discriminant divisor D has normal crossings, and

that there is a J morphism $J : B \to P^1(B)$. If two or more components of the discriminant divisor meet at a point $p \epsilon B$ then the reduction types of these components lie in the same group.

15.3 Pre-Settled

We have already mentioned that an elliptic scheme with a J morphism is automatically settled. We show now that it is pre-settled. To to this, we return to the definition of the straight part of the discriminant. It is computed for each reduction type in chart 141.

We notice that for each reduction type with $J \neq \infty$ we have the simple relation

$$straight(X) = v(d). \tag{211}$$

and for the reduction types with $J = \infty$ we have the simple relation

$$straight(X) = 0. \tag{212}$$

Now suppose we blow up the base: $B' \to B$, and let $p \epsilon B$. The J morphism clearly pulls back to B'. Suppose p passes through two or more components of the discriminant locus. Then by proposition 15.3 we know that the type of the exceptional divisor is the same group as the types involved in the collision.

Assume $J(p) = \infty$. Then $residual_i(p) = 0$ for each component in the discriminant divisor and also $residual_e(p) = 0$. Thus the pre-settled fault at p is 0.

Assume $J(p) \neq \infty$. Then $residual_i(p) = v_i(d)$ for each component in the discriminant divisor and also $residual_e(p) = v_e(d)$. But we also know that the discriminant has normal crossings so by equation 137 its valuation is additive in blow ups. Thus the pre-settled fault at p is also 0.

These two cases show that an elliptic scheme with a J morphism is a pre-settled elliptic scheme.

Proposition 15.4 (J Morphism Implies Pre-Settled) *Let B be a regular Noetherian n - dimensional integral separated scheme. Suppose that none of the residue fields of B have residue characteristic 2 or 3. Let $X \to B$ be an elliptic subscheme of $P^2(B)$ defined locally by Weierstrass Equations. Suppose the discriminant divisor has normal crossings, and there exists a J morphism*

$$J : B \to P^1(B)$$

extending the J invariant for non-singular elliptic curves. Then the elliptic scheme is settled and pre-settled.

This now allows us to apply the main theorem 1.2 to conclude that the scheme $X \to B$ does have a flat resolution. Thus we have proven theorem 1.3.

15.4 Modular Morphism

One may interpret a J morphism as a morphism from B to the full modular curve $X(1)$. If we were to strengthen this assumption, and hypothesize that there was a morphism

$$B \to X(n) \tag{213}$$

for $n \geq 3$ then the only possible reduction types are I_0, and I_n, so the analysis trivializes in this case.

15.5 Constructing J

It is possible to eliminate the assumption of the J morphism, and assume instead that c_4 and c_6 have normal crossings. (I do not use this, elsewhere in the thesis)

Using chart 205, one can prove that if the c_4, c_6, and d have normal crossings then the J map already extends to a regular map.

Sketch of proof: Suppose a types I_n collides with a second type II, III, IV, IV^*, III^*, or II^*. Consider blowing up the intersection of the two types.

Since $v_1(c_4) = v_1(c_6) = 0$, The exceptional valuation is as

$$v_s(c_4) = v_2(c_4)$$

$$v_s(c_6) = v_2(c_6)$$

Thus the reduction type of the exceptional divisor is the same as the reduction type of the second type. This is a contradiction since we know $v_s(d) > v_2(d)$. The same argument shows that a type I_n^* may not collide with one of the above types.

Now assume that a type III or III^* collides with a second type II, IV, IV^*, or II^*. Consider blowing up the intersection of the two types. By the above argument, the exceptional divisor can not be a type I_n or I_n^* with $n > 0$. But since the discriminant divisor has normal crossings, we conclude that $v_s(d) \epsilon \{1, 5, 7, 11\}$. This can not happen.

Thus we conclude that types from separate groups may not collide. This will imply pre-settled as in the previous subsection. However, it also proves that the ration map J, is already a morphism.

16 The Collisions

16.1 Away From 2 and 3

If we limit ourselves to the study of elliptic schemes X over bases B with no points of residue characteristic 2 or 3, the analysis in this paper simplifies, but we still obtain very interesting fibers in a regular model $X'' \to B$. In fact, most of the special fibers that we discover in the collisions over more general bases have the same structure as those which appear over these limited bases.

In order to preview the special fibers that arise in our regular model, we first summarize the special fibers which appear in a regular model over a base where $1/6$ is in all of the local rings of B. We describe these special fibers in a proposition and with a chart of collisions.

Theorem 16.1 (Collisions Away from 2 and 3)
Let B be a regular Noetherian n-dimensional integral separated scheme, Let $X \to B$ be a limited Weierstrass elliptic scheme defined over B, and suppose B has no points of residue characteristic 2 or 3. Then there exists a series of blow ups $X'' \to X$ such that X'' is regular, minimal, projective and flat over B.

Over points of B not on the discriminant divisor the fibers of the map $X'' \to B$ are non-singular elliptic curves.

Over non-singular points of the discriminant divisor the fibers of the map $X'' \to B$ are the reduction types on Kodaira's list [KOD].

The only types of collisions that occur between reduction types are as in chart 214, and over these singular points of the discriminant divisor the fibers of the map $X'' \to B$ are also given by chart 214.

$$
\begin{array}{ccc}
\underline{Types\,in\,Collision} & & \underline{Special\,Fiber} \\
II+I_0^* & & 123 \\
II+IV^* & & 12342 \\
IV+I_0^* & & 1232 \\
III+I_0^* & & 12321 \\
I_n+I_m & & I_{n+m} \\
I_n+I_m^* & (n-odd) & I^+_{(n-1)/2+m} \\
I_n+I_m^* & (n-even) & I^*_{n/2+m}
\end{array}
\tag{214}
$$

The special fibers appearing in the last column of chart 214 consist of various rational curves of given multiplicities intersection transversally. They will be described in section 16.6.

The I_k^+ type consists of 2 multiplicity 1 components connected to a chain of $k+2$ multiplicity 2 components. This is similar to a type $I_k{}^*$, which has $k+1$ multiplicity 2 components, but a pair of final multiplicity 1 components. Thus type I_k^+ looks like type I_k^* with the final two components identified. This type will be constructed in section 16.8.

In the collisions specified by one of the last three lines in chart 214 there can be any number of I_n types present in the collision, up to the dimension of the base scheme. However, there may be at most one I_n type with n odd. If there are multiple I_n types colliding at a point, the special fiber is still given by chart 214, with n replaced by $\sum n_i$.

We do not prove theorem 16.1 here because it will be a corollary of the main theorem 1.2 of this paper, and the description of the fibers in the general case.

16.2 The Regular Model Exists

The main theorem 1.2, which the rest of the paper proves, depends on a desingularization of a limited Weierstrass elliptic scheme. This desingularization is the content of the following theorem.

Theorem 16.2 (Regular Model Exists)

Let B be a regular Noetherian n-dimensional integral separated scheme, Let $X \to B$ be a limited Weierstrass elliptic scheme defined over B. Then there exists a series of blow ups $X'' \to X$ such that X'' is regular, minimal, projective and flat over B.

We will also describe all fibers of the morphism $X'' \to B$ in section 17. In particular we will describe the collision fibers in chart 310.

16.3 Construction of the Regular Model

16.3.1 Applying Tate's Algorithm

To prove theorem 16.2 we let $X \to B$ be a limited Weierstrass elliptic scheme. Not only is B settled and pre-settled, but X has only the types of collisions as specified in section 14. For each component of the discriminant divisor we now blow up X as prescribed by theorem 7.1, the extended Tate's algorithm. The blow ups of Tate's algorithm can be found in [TA] and [SIL 2], and we explicitly described the blow ups for the new reduction types in section 6. Fortunately a series of such blow ups does produce a regular scheme.

Suppose T is a component of the discriminant divisor defined locally by $t = 0$. We define blow ups of X a-priori also locally by ideals such as (t, x, y) or (x, y). Although x and y depend on a local Weierstrass equation, the fact that each Weierstrass equation is in chart form implies that these local ideals define a closed subscheme of X. This just means the ideals in the local rings patch together to form a sheaf of ideals on X that define the blow up.

For example, in the first blow up in Tate's algorithm, the ideals locally defined by (t, x, y) define the subscheme of X that passes through the singular point of each fiber over the component T.

16.3.2 The Blow ups of X

We thus define the blow ups globally by indicating over which components of the discriminant divisor we perform Tate's algorithm. We later then analyze the results locally to check regularity.

To construct the regular model X'', consider the set of components of the discriminant divisor. This set is finite and each component has a reduction type which is computable by theorem 7.1.

First perform the blow-ups specified in the extended Tate's algorithm for each component of type II^*. Then do this for each component of type III^*, etc. We continue this process for a specific order of the reduction types. Let us formalize the algorithm.

Algorithm 16.3 *Let X be a limited Weierstrass elliptic scheme. Let S be the set of the components of the discriminant divisor. Choose an order on the set S such that $T_1 < T_2$ if type T_1 appears before T_2 on the following list.*

$$II^*, III^*, IV^*, Y3, I_0^*, X2, IV, Y2, III, II, I_n^*, K_n, K_n', X1, Y1, Z1 \tag{215}$$

Then for this order of S, sequentially blow up X for each component $T \epsilon S$ according to the extended Tate's algorithm.

Note that this is not the only order which will work, but this order suffices to desingularize X. We also remark that types $X1, Y1, Z1, K1$, and II actually require no blow ups, so they can just as well be omitted from the list 215.

16.3.3 Describing X''

We have now constructed a scheme X'' and a morphism $X'' \to X$.

The scheme X'' is projective because by the definition of blow ups, X'' will be a subscheme of a product of projective spaces. By the Segre embedding X'' is also a subscheme of of $P^N(B)$ for some large N.

X'' will be minimal in the sense that it the blow up of a limited Weierstrass elliptic scheme X as defined in section 8.6. That is all, all localized Weierstrass equations over the DVRs of the components of the discriminant divisor are minimal Weierstrass equations over the DVR.

Suppose we know that X'' is a regular scheme. Because B is regular, the criteria for flatness is just a check on the fiber dimension of $X \to B$. Once we show that every fiber is of pure dimension one, we will know that X is flat over B. This criteria is a case of theorem 23.1 in [MAT].

We now show that X is a regular scheme. To do this we analyze the results of the blow ups of algorithm 16.3 locally. The scheme X is trivially regular at all points not on a fiber of the discriminant locus. Suppose first that $p \epsilon B$ belongs to only one component of the discriminant locus. The analysis of such points differs little from the analysis of elliptic schemes over DVRs, and it is discussed briefly in section 16.4.

Now suppose that $p \epsilon B$ belongs to more than one component of the discriminant locus, so that p is a collision point. We show that every point of X'' in the fiber above p is a regular point of X'' by analyzing the results of algorithm 16.3 over local rings O_p. Depending on the broad type of collision, these results are analyzed in sections 16.6, 16.7, or 16.8.

When we have shown that every point of X'' is regular, we will have finished the proof of theorem 16.2, and thus also of the main theorem of the paper, theorem 1.2. However, we also prove more. In addition to checking regularity in the following subsections, we also describe all of the special fibers geometrically.

16.4 The Simple Fibers

We first discuss the fibers of $X'' \to B$ at points p that belong to only one component of the discriminant divisor. These fibers are on Kodaira's list [KOD], or are one of the new fiber types described in section 7.2. In the proof of Tate's algorithm [TA] and in the extension to it 7.1 regularity is proved by showing that after dividing out by appropriate powers of the t_i, one of the a_j is a unit in O_p. That is, we use the special form of a_j to show that the cotangent space $\frac{m}{m^2}$ at every point in the special fiber has the correct dimension. In particular this was done in section 6, when computing the new fiber types.

The same arguments hold when B is not a DVR, with a slight modification. We do not reproduce these calculations in full here, but only mention that the settled and pre-settled hypothesis force the equivalent condition of the coefficient a_j. In this case we consider the morphisms ψ, ϕ_1, and ϕ_2 at a point p where X is given by a Weierstrass equation in multiple chart form. One or more of the coefficients a_j are forced to be $\{t_i\}$ normal elements. We can consult charts 153, 154, 187, or section 12.5.1 to determine which coefficients a_j are indeed $\{t_i\}$ normal elements.

16.5 The Collision Fibers

In this subsection we assume that $p \epsilon B$ is a collision point. Because we take X to be a limited Weierstrass elliptic scheme, the collision must be between reduction types as specified in propositions 14.4, 14.5, 14.6, and 14.7. In addition to specifying which reduction types can participate in the collision, these propositions also specify which coefficients a_i are forced to be $\{t_i\}$ normal elements. This special form of this coefficient will be used to show regularity at points of X in the fiber over p.

As defined in 13.4, the reduction types in a collision are either zero types or main types. The collision types that exists in a limited Weierstrass elliptic scheme can be grouped into three broad categories.

If a collision involves two main types plus some possible zero types, we call it a Double Type Collision and consider it in section 16.6. If a collision involves one main types plus one or more zero types, but no types in an infinite family of reduction types, we call it a Single Type Collision and consider it in section 16.7. If the collision does involve a type in an infinite family of reduction types, we call it a Multiple Type Collision and consider it in section 16.8. In this third category even the zero types (such as I_n) require a series of blow ups.

Our multidimensional version of Tate's algorithm, algorithm 16.3, is composed of a series of blow ups, and to fully analyze a scheme defined by a blow up, more than one coordinate patch must be examined. We have already seen an example of multiple coordinate patched in section 9.2.1 when blowing up a regular ring. For brevity, in the computations which follow, not all coordinate patches will be shown. Rather, we only display the coordinate patches where new components of the special fiber arise. We also remark that a singular point of X'' must be a singular point of the fiber over p. This limits the number of points that we must check for regularity.

16.6 Double Type Collisions

Here we consider collisions at $p \epsilon B$ that involve two main types plus some possible zero types. These collisions are described in propositions 14.4 and 14.6.

The limited Weierstrass elliptic scheme at p is described by a Weierstrass equation in multiple chart form with coefficients in the local ring O_p. Let $(s, t, r_1 ... r_k)$ be a discriminant compatible set of uniformizing parameters at O_p such that $s = 0$ and $t = 0$ define the two main types involved in the collision. For some i, a zero type may be defined by some $r_i = 0$.

As specified in propositions 14.4 and 14.6 these collision must have modulus 4 or 6, and furthermore either a_4 or a_6 must be a $\{t_i\}$ normal element.

We are going to analyze the effect at p of the blow ups of algorithm 16.3 that construct X'' from X. After each blow up, we examine the fibers and check for regularity. In case the residue characteristic of O_p is not 2, we simplify the following calculations by setting $a_1 = 0$ and $a_3 = 0$. In the computations that follow let κ_s, and κ_t be the residue fields of the DVRs obtained by localizing O_p at $s = 0$ and $t = 0$, respectively. We also denote the residue field of O_p by κ.

16.6.1 The A6 (1,3) Collision

In this subsection we assume that the modulus of the collision is 6, and that $v_s(a_6) = 1$, and $v_t(a_6) = 3$. In this case we also know $v_{r_i}(a_6) = 0$ for the other uniformizing parameters r_i, and that a_6 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 216 describes what the two main types and possible zero types in the collision can be based on the residue characteristic of O_p.

$\underline{Residue\,Char}$	$\underline{Main\,Type\,s}$	$\underline{Main\,Type\,t}$	$\underline{Zero\,Types}$	
$\neq 2,3$	II	I_0^*	$-$	
3	II	$I_0^*, Z2$	$Z1$	(216)
2	II	$I_0^*, X2$	$X1, Y1$	

We group these cases together, since the Weierstrass equation has a similar form in each case, and the same blow ups are required in each case.

We start with the subscheme of $O_p[x,y]$ defined by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{217}$$

with the valuations of the a_i as follows.

	a_1	a_2	a_3	a_4	a_6	
v_s	≥ 1	≥ 1	≥ 1	≥ 1	$= 1$	(218)
v_t	≥ 1	≥ 1	≥ 2	≥ 2	$= 3$	

We also assume that the polynomial

$$X^3 + \frac{a_2}{t}X^2 + \frac{a_4}{t^2}X + \frac{a_6}{t^3} \tag{219}$$

has no double or triple roots that are defined over κ_t.

Since a_6 has normal crossings there is a unit u so that

$$a_6 = ust^3. \tag{220}$$

We now perform the blow ups prescribed by algorithm 16.3. These are just the blow ups in Tate's algorithms for the two components $s = 0$, and $t = 0$. We compute the special fiber in κ by setting $s = t = r_i = 0$.

The beginning special fiber is a rational curve of multiplicity 1 given by $y^2 = x^3$. All points of X'' on this component except the point $x = y = 0$ are regular.

Tate's algorithm specifies no blow ups over $s = 0$, but two over $t = 0$. First blow up at the ideal (x, y, t). For the third coordinate patch put $x = x_1 t$ and $y = y_1 t$. This patch is the affine subscheme of $O_p[x_1, y_1]$ defined by

$$y_1^2 + a_1 x_1 y_1 + \frac{a_3}{t} y_1 = x_1^3 t + a_2 x_1^2 + \frac{a_4}{t} x_1 + \frac{a_6}{t^2}. \tag{221}$$

The special fiber is the multiplicity 2 rational curve defined by $y_1^2 = 0$. All points of X'' on this component except $x_1 = y_1 = 0$ are regular.

Next blow up at the ideal (y_1, t). For the second coordinate patch put $y_1 = y_2 t$. This patch is the affine subscheme of $O_p[x_1, y_2]$ defined by

$$y_2^2 t + a_1 x_1 y_2 + \frac{a_3}{t} y_2 = x_1^3 + \frac{a_2}{t} x_1^2 + \frac{a_4}{t^2} x_1 + \frac{a_6}{t^3}. \tag{222}$$

The special fiber is the multiplicity 3 rational curve defined by $x_1^3 = 0$.

This last equation modulo the square of (s, t, x_1), is

$$y_2^2 t = u\, s \tag{223}$$

for some unit $u \epsilon O_p$. Thus all points of X'' on this component are regular.

Indeed every point of X'' above p is regular, and the complete special fiber is a chain of rational curves of multiplicity 1, 2 and 3.

Note that the presence of one or more zero types does not affect the computation of the special fiber or the regularity at points on the fiber. This is because the zero types $X1$, $Y1$, and $Z1$, which may be present in the collision all have $v_{r_i}(a_6) = 0$. This concludes the computation of the special fiber and proof of regularity for this case.

16.6.2 The A6 (2,3) Collision

In this subsection we assume that the modulus of the collision is 6, and that $v_s(a_6) = 2$, and $v_t(a_6) = 3$. In this case we also know $v_{r_i}(a_6) = 0$ for the other uniformizing parameters r_i, and that a_6 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 224 describes what the two main types and possible zero types in the collision can be based on the residue characteristic of O_p.

$\underline{Residue\,Char}$	$\underline{Main\,Type\,s}$	$\underline{Main\,Type\,t}$	$\underline{Zero\,Types}$
$\neq 2,3$	IV	I_0^*	$-$
3	IV	$I_0^*, Z2$	$Z1$
2	$IV, Y2$	$I_0^*, X2$	$X1, Y1$

(224)

We group these cases together, since the Weierstrass equation has a similar form in each case, and the same blow ups are required in each case.

We start with the subscheme of $O_p[x, y]$ defined by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{225}$$

with the valuations of the a_i as follows.

	a_1	a_2	a_3	a_4	a_6
v_s	≥ 1	≥ 1	≥ 1	≥ 2	$= 2$
v_t	≥ 1	≥ 1	≥ 2	≥ 2	$= 3$

(226)

We also assume that the polynomial

$$X^2 + \frac{a_3}{s}X + \frac{a_6}{s^2} \tag{227}$$

has no double root that is defined over κ_s. and that the polynomial

$$X^3 + \frac{a_2}{t}X^2 + \frac{a_4}{t^2}X + \frac{a_6}{t^3} \tag{228}$$

has no double or triple roots that are defined over κ_t.

Since a_6 has normal crossings there is a unit u so that

$$a_6 = us^2t^3. \tag{229}$$

We now perform the blow ups prescribed by algorithm 16.3. These are just the blow ups in Tate's algorithms for the two components $s = 0$, and $t = 0$. We compute the special fiber in κ by setting $s = t = r_i = 0$.

The beginning special fiber is a rational curve of multiplicity 1 given by $y^2 = x^3$. All points of X'' on this component except the point $x = y = 0$ are regular.

Tate's algorithm specifies one blow up over $s = 0$, and two over $t = 0$. First blow up at the ideal (x, y, t). For the third coordinate patch put $x = x_1 t$ and $y = y_1 t$. This patch is the affine subscheme of $O_p[x_1, y_1]$ defined by

$$y_1^2 + a_1 x_1 y_1 + \frac{a_3}{t} y_1 = x_1^3 t + a_2 x_1^2 + \frac{a_4}{t} x_1 + \frac{a_6}{t^2}. \tag{230}$$

The special fiber is the multiplicity 2 rational curve defined by $y_1^2 = 0$. All points of X'' on this component except $x_1 = y_1 = 0$ are regular.

Next blow up at the ideal (y_1, t). For the second coordinate patch put $y_1 = y_2 t$. This patch is the affine subscheme of $O_p[x_1, y_2]$ defined by

$$y_2^2 t + a_1 x_1 y_2 + \frac{a_3}{t} y_2 = x_1^3 + \frac{a_2}{t} x_1^2 + \frac{a_4}{t^2} x_1 + \frac{a_6}{t^3}. \tag{231}$$

The special fiber is the multiplicity 3 rational curve defined by $x_1^3 = 0$.

This last equation modulo the square of (s, t, x_1), is

$$y_2^2 t = 0. \tag{232}$$

Thus all points of X'' on this component except the point $x_1 = y_2 = 0$ are regular.

Next blow up at the ideal (x_1, y_2, s). For the second coordinate patch put $x_1 = x_2 y_2$ and $s = y_2 g$. This patch is an affine subscheme of $O_p[x_2, y_2, g]$ defined by two equations.

$$t + a_1 x_2 + \frac{a_3}{st} g = x_2^3 y_2 + \frac{a_2}{t} x_2^2 + \frac{a_4}{st^2} x_2 g + \frac{a_6}{s^2 t^3} g^2. \tag{233}$$

$$s = y_2 g. \tag{234}$$

The special fiber is defined by the equations $y_2 g = 0$, and $x_2^3 y_2 + \frac{a_6}{s^2 t^3} g^2$, and has two components. The $g = x_2 = 0$ component has multiplicity 3, and has already appeared before this blow up. The exceptional divisor of this blow up is the $g = y_2 = 0$ component.

At points where the two components meet, $x_2 = y_2 = 0$. Since equations 233 and 234 modulo the square of (s, t, x_2, y_2) are $s = 0$, and $t = 0$, the scheme is not singular at that point.

Focusing on the new component, we consider the open subscheme where $x_2 \neq 0$. Multiplying $s = y_2 g$ by x_2^3 and eliminating y_2 in equation 233, shows us that this open subscheme is isomorphic to the subscheme of $O_p[x_2, g]$ defined by a single equation.

$$g(t + a_1 x_2 + \frac{a_3}{st} g) = x_2^3 s + g(\frac{a_2}{t} x_2^2 + \frac{a_4}{st^2} x_2 g + \frac{a_6}{s^2 t^3} g^2) g. \qquad (235)$$

The special fiber is defined by $\frac{a_6}{s^2 t^3} g^3 = 0$. But $\frac{a_6}{s^2 t^3}$ is a unit so the special fiber is just the multiplicity 3 rational curve defined by $g^3 = 0$.

Equation 235 modulo the square of (s,t,g) is

$$x_2^3 s = 0. \qquad (236)$$

Because on this open subscheme $x_2 \neq 0$, this shows that every point of X'' on this component is regular.

Indeed every point of X'' above p is regular, and the complete special fiber is a chain of rational curves of multiplicity 1, 2, 3 and 3.

Note that the presence of one or more zero types does not affect the computation of the special fiber or the regularity at points on the fiber. This is because the zero types $X1$, $Y1$, and $Z1$, which may be present in the collision all have $v_{r_i}(a_6) = 0$. This concludes the computation of the special fiber and proof of regularity for this case.

16.6.3 The A6 (1,4) Collision

In this subsection we assume that the modulus of the collision is 6, and that $v_s(a_6) = 1$, and $v_t(a_6) = 4$. In this case we also know $v_{r_i}(a_6) = 0$ for the other uniformizing parameters r_i, and that a_6 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 237 describes what the two main types and possible zero types in the collision can be based on the residue characteristic of O_p.

Residue Char	Main Type s	Main Type t	Zero Types	
$\neq 2, 3$	II	IV^*	$-$	
3	II	IV^*	$Z1$	(237)
2	II	$IV^*, Y3$	$X1, Y1$	

We group these cases together, since the Weierstrass equation has a similar form in each case, and the same blow ups are required in each case.

We start with the subscheme of $O_p[x, y]$ defined by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{238}$$

with the valuations of the a_i as follows.

	a_1	a_2	a_3	a_4	a_6	
v_s	≥ 1	≥ 1	≥ 1	≥ 1	$= 1$	(239)
v_t	≥ 1	≥ 2	≥ 2	≥ 3	$= 4$	

We also assume that the polynomial

$$X^3 + \frac{a_2}{t}X^2 + \frac{a_4}{t^2}X + \frac{a_6}{t^3} \tag{240}$$

has no double root that is defined over κ_t.

Since a_6 has normal crossings there is a unit u so that

$$a_6 = ust^4. \tag{241}$$

We now perform the blow ups prescribed by algorithm 16.3. These are just the blow ups in Tate's algorithms for the two components $s = 0$, and $t = 0$. We compute the special fiber in κ by setting $s = t = r_i = 0$.

The beginning special fiber is a rational curve of multiplicity 1 given by $y^2 = x^3$. All points of X'' on this component except the point $x = y = 0$ are regular.

Tate's algorithm specifies no blow ups over $s = 0$, but three over $t = 0$. First blow up at the ideal (x, y, t). For the third coordinate patch put $x = x_1 t$ and $y = y_1 t$. This patch is the affine subscheme of $O_p[x_1, y_1]$ defined by

$$y_1^2 + a_1 x_1 y_1 + \frac{a_3}{t} y_1 = x_1^3 t + a_2 x_1^2 + \frac{a_4}{t} x_1 + \frac{a_6}{t^2}. \tag{242}$$

The special fiber is the multiplicity 2 rational curve defined by $y_1^2 = 0$. All points of X'' on this component except $x_1 = y_1 = 0$ are regular.

Next blow up at the ideal (y_1, t). For the second coordinate patch put $y_1 = y_2 t$. This patch is the affine subscheme of $O_p[x_1, y_2]$ defined by

$$y_2^2 t + a_1 x_1 y_2 + \frac{a_3}{t} y_2 = x_1^3 + \frac{a_2}{t} x_1^2 + \frac{a_4}{t^2} x_1 + \frac{a_6}{t^3}. \tag{243}$$

The special fiber is the multiplicity 3 rational curve defined by $x_1^3 = 0$.

This last equation modulo the square of (s, t, x_1), is

$$y_2^2 t = 0. \tag{244}$$

Thus all points of X'' on this component except the point $x_1 = y_2 = 0$ are regular.

The next blow up prescribed by Tate's algorithm is at the ideal $(x_1, y_2^2 - \frac{a_6}{t^4}, t)$. Note that $\frac{a_6}{t^4}$ is not a square in O_p modulo t. We blow up at this ideal, and for brevity we will skip the second and third coordinate patches.

For the first coordinate patch put $x_1 b = y_2^2 - \frac{a_6}{t^4}$, and $x_1 c = t$. Then this patch is an affine subscheme of $O_p[x_1, y_2, b, c]$ defined by three equations.

$$x_1 b = y_2^2 - \frac{a_6}{t^4}. \tag{245}$$

$$x_1 c = t. \tag{246}$$

$$bc + \frac{a_1}{t} c y_2 + \frac{a_3}{t^3} y_2 c^2 = x_1 + \frac{a_2}{t} + \frac{a_4}{t^3} c. \tag{247}$$

The last equation may be solved for x, so we can eliminate it.

In particular, by setting $s = t = 0$, we compute that the special fiber is defined by the equations $y_2^2 = 0$,and $bc^2 = 0$. The $c = y_2 = 0$ component has multiplicity 4, and the $b = y_2 = 0$ component has multiplicity 2.

To show regularity requires that we compute some derivatives. By eliminating x_1 from equations 245, 246, and 247, we see that this patch is isomorphic to a subscheme of $O_p[y_2, b, c]$ given by two equations.

$$(bc + \frac{a_1}{t}cy_2 + \frac{a_3}{t^3}y_2c^2)c = t + (\frac{a_2}{t} + \frac{a_4}{t^3}c)c. \quad (248)$$

$$(bc + \frac{a_1}{t}cy_2 + \frac{a_3}{t^3}y_2c^2)b = y_2^2 - \frac{a_6}{t^4} + (\frac{a_2}{t} + \frac{a_4}{t^3}c)b. \quad (249)$$

Call the two equations f and g. At points with $s = t = y_2 = c = 0$, the matrix of partial derivatives is given by

$$\begin{bmatrix} \frac{df}{ds} & \frac{df}{dt} \\ \frac{dg}{ds} & \frac{dg}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{a_6}{st^4} & 0 \end{bmatrix}. \quad (250)$$

At points with $s = t = y_2 = b = 0$, the matrix of partial derivatives is given by

$$\begin{bmatrix} \frac{df}{ds} & \frac{df}{dt} \\ \frac{dg}{ds} & \frac{dg}{dt} \end{bmatrix} = \begin{bmatrix} * & 1 \\ \frac{a_6}{st^4} & 0 \end{bmatrix}. \quad (251)$$

On the special fiber either $b = 0$, or $c = 0$, and in each case the matrix of partial derivatives is nonsingular, so X'' is regular at all points on these two components.

Indeed every point of X'' above p is regular, and the complete special fiber is a chain of rational curves of multiplicity 1, 2, 3, 4, and 2.

Note that the presence of one or more zero types does not affect the computation of the special fiber or the regularity at points on the fiber. This is because the zero types $X1$, $Y1$, and $Z1$, which may be present in the collision all have $v_{r_i}(a_6) = 0$. This concludes the computation of the special fiber and proof of regularity for this case.

16.6.4 The A4 (1,2) Collision

In this subsection we assume that the modulus of the collision is 4, and that $v_s(a_4) = 1$, and $v_t(a_4) = 2$. In this case we also know $v_{r_i}(a_4) = 0$ for the other uniformizing parameters r_i, and that a_4 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 252 describes what the two main types and possible zero types in the collision can be based on the residue characteristic of O_p.

$Residue\,Char$	$Main\,Type\,s$	$Main\,Type\,t$	$Zero\,Types$
$\neq 2,3$	III	I_0^*	$-$
3	III	I_0^*	$-$
2	III	$I_0^*, X2$	$X1$

(252)

We group these cases together, since the Weierstrass equation has a similar form in each case, and the same blow ups are required in each case.

We start with the subscheme of $O_p[x, y]$ defined by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{253}$$

with the valuations of the a_i as follows.

	a_1	a_2	a_3	a_4	a_6
v_s	≥ 1	≥ 1	≥ 1	$= 1$	≥ 2
v_t	≥ 1	≥ 1	≥ 2	$= 2$	≥ 3

(254)

We also assume that the polynomial

$$X^3 + \frac{a_2}{t}X^2 + \frac{a_4}{t^2}X + \frac{a_6}{t^3} \tag{255}$$

has no double or triple roots that are defined over κ_t.

Since a_4 has normal crossings there is a unit u so that

$$a_4 = ust^2. \tag{256}$$

We now perform the blow ups prescribed by algorithm 16.3. These are just the blow ups in Tate's algorithms for the two components $s = 0$, and $t = 0$. We compute the special fiber in κ by setting $s = t = r_i = 0$.

The beginning special fiber is a rational curve of multiplicity 1 given by $y^2 = x^3$. All points of X'' on this component except the point $x = y = 0$ are regular.

Tate's algorithm specifies one blow up over $s = 0$, and two over $t = 0$. First blow up at the ideal (x, y, t). For the third coordinate patch put $x = x_1 t$ and $y = y_1 t$. This patch is the affine subscheme of $O_p[x_1, y_1]$ defined by

$$y_1^2 + a_1 x_1 y_1 + \frac{a_3}{t} y_1 = x_1^3 t + a_2 x_1^2 + \frac{a_4}{t} x_1 + \frac{a_6}{t^2}. \tag{257}$$

The special fiber is the multiplicity 2 rational curve defined by $y_1^2 = 0$. All points of X'' on this component except $x_1 = y_1 = 0$ are regular.

Next blow up at the ideal (y_1, t). For the second coordinate patch put $y_1 = y_2 t$. This patch is the affine subscheme of $O_p[x_1, y_2]$ defined by

$$y_2^2 t + a_1 x_1 y_2 + \frac{a_3}{t} y_2 = x_1^3 + \frac{a_2}{t} x_1^2 + \frac{a_4}{t^2} x_1 + \frac{a_6}{t^3}. \tag{258}$$

The special fiber is the multiplicity 3 rational curve defined by $x_1^3 = 0$.

This last equation modulo the square of (s, t, x_1), is

$$y_2^2 t = 0. \tag{259}$$

Thus all points of X'' on this component except the point $x_1 = y_2 = 0$ are regular.

Next blow up at the ideal (x_1, y_2, s). For the second coordinate patch put $x_1 = x_2 y_2$ and $s = y_2 g$. This patch is an affine subscheme of $O_p[x_2, y_2, g]$ defined by two equations.

$$t + a_1 x_2 + \frac{a_3}{st} g = x_2^3 y_2 + \frac{a_2}{t} x_2^2 + \frac{a_4}{st^2} x_2 g + \frac{a_6}{s^2 t^3} g^2. \tag{260}$$

$$s = y_2 g. \tag{261}$$

The special fiber is defined by two equations.

$$x_2^3 y_2 + \frac{a_4}{st^2} x_2 g + \frac{a_6}{s^2 t^3} g^2. \tag{262}$$

$$y_2 g = 0. \tag{263}$$

There are three components in this fiber. The first component is the multiplicity 3 rational curve defined by $g = x_2 = 0$. It has already appeared before this blow up.. The second component is the multiplicity 2 rational curve defined by $g = y_2 = 0$. The third component is the multiplicity 1 rational curve and is defined by $y_2 = 0$ and

$$\frac{a_4}{st^2}x_2 + \frac{a_6}{s^2t^3}g. \tag{264}$$

These three components meet at the point $g = x_2 = y_2 = 0$. The equations 262 and 263 modulo the ideal (s, t, g, x_2, y_2) are just $t = 0$ and $s = 0$, so this point of X'' is a regular point.

In fact we can easily see that X'' is regular at all points in this coordinate patch with $g = 0$. Call the two equations f and g. At points with $s = t = g = 0$, the matrix of partial derivatives is given by

$$\begin{bmatrix} \frac{df}{ds} & \frac{df}{dt} \\ \frac{dg}{ds} & \frac{dg}{dt} \end{bmatrix} = \begin{bmatrix} \frac{da_2}{ds} & 1 \\ 1 & 0 \end{bmatrix}. \tag{265}$$

To be complete we could check the other coordinate patches for regularity. We omit this check and just state that indeed every point of X'' above p is regular, and the complete special fiber is a chain of rational curves of multiplicity 1, 2, 3, 2 and 1, such that the last three components meet in a point.

Note that the presence of one or more zero types does not affect the computation of the special fiber or the regularity at points on the fiber. This is because the zero types $X1$, $Y1$, and $Z1$, which may be present in the collision all have $v_{r_i}(a_6) = 0$. This concludes the computation of the special fiber and proof of regularity for this case.

16.6.5 The A4 (1,2) Alternate Collision

In this subsection we present a variation of the computation of section 16.6.4. The purpose of this section is to highlight the fact that the regular model X'' is not unique. In particular, the order of the blow ups is relevant to the special fiber types that arise.

Section 16.6.4 relies of the fact that in algorithm 16.3, Tate's algorithm is computed first for the main type I_0^* or $X2$, and secondly for the main type III. Suppose instead, we alter the order of algorithm 16.3 such that type III is the first type to appear in list 215. Then the collision of section 16.6.4 is the only one which would be altered. I briefly discuss the blow ups and results at such a collision point.

The first component of the special fiber is a multiplicity 1 rational curve. Blow up at the ideal (x, y, s), and examine the third coordinate patch. This component is a multiplicity 2 rational curve. Blow up at the ideal (x, y, t), and examine the first and third coordinate patch. In the first patch there is a new multiplicity 2 rational curve intersecting the previous one. The point of intersection is a singular point of X''. In the third patch there is just the same new multiplicity 2 rational but with a singular point that does not appear in the first patch. Blow up along the whole new multiplicity two component. In patch one of the previous blow up the exceptional divisor is a multiplicity 3 component. In patch three of the previous blow up the exceptional divisor is a multiplicity 1 component.

It is easy to check that every point of X'' above p is regular, and the complete special fiber is also a chain of rational curves of multiplicity 1, 2, 3, 2 and 1. However in contrast with the results of section 16.6.4, the last multiplicity one component does not meet the multiplicity two component.

We conclude that the order of blow ups over the two discriminant divisors does, in general, matter. In this case the special fibers, although similar do have different geometry. The fibers are indeed very similar, we will still call them both chains of rational curves of multiplicity 1, 2, 3, 2 and 1.

16.6.6 Summary of Double Collision Fibers

Proposition 16.4 (Double Fiber Possibilities)
Let X'' be the regular model of a limited Weierstrass elliptic scheme over a base B, constructed by algorithm 16.3. Let $p \epsilon B$, and let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters at p. Suppose the reduction types T_1 and T_2 are main type and all other types are zero types.

Suppose the modulus of the collision is 6, and that T_1 is type II and type T_2 is type I_0^*, type $X2$ or type $Z2$. Then the special fiber of X'' at p is a chain of rational curves with multiplicities

$$1 - 2 - 3. \tag{266}$$

Suppose the modulus of the collision is 6, and that T_1 is type II and type T_2 is type IV^*, or type $Y3$. Then the special fiber of X'' at p is a chain of rational curves with multiplicities

$$1 - 2 - 3 - 4 - 2. \tag{267}$$

Suppose the modulus of the collision is 6, and that T_1 is type IV and type T_2 is type I_0^*, type $X2$ type $Z2$. Then the special fiber of X'' at p is a chain of rational curves with multiplicities

$$1 - 2 - 3 - 2. \tag{268}$$

Suppose the modulus of the collision is 4, and that T_1 is type III and type T_2 is type I_0^* or type $X2$. Then the special fiber of X'' at p is a chain of rational curves with multiplicities

$$1 - 2 - 3 - 2 - 1. \tag{269}$$

The last multiplicity 1 component also meets the multiplicity 3 component.

There are no collisions involving two main types except those described above.

Refer to section 16.6.5 for an alternate fiber type of the modulus 4 collision described in proposition 16.4.

16.7 Single Type Collisions

Here we consider collisions that involve one main types plus at least one zero type, but no types in an infinite family of reduction types. The possible zero types are $X1$, $Y1$, and $Z1$. These types of collisions only occur at points p with residue characteristic 2 or 3.

Such collisions will be described by a Weierstrass equation in multiple chart form with coefficients in the local ring O_p. Let $(t, r_1 ... r_k)$ be a discriminant compatible set of uniformizing parameters at O_p such that $t = 0$ defines the two main type involved in the collision. For at least one i, a zero type is defined by $r_i = 0$.

The colliding types are described in propositions 14.5 and 14.7, and the zero types must be among $X1$, $Y1$, and $Z1$. These reduction types do not require any blow ups, so the only blow ups are those prescribed by Tate's algorithm for the main type defined by $t = 0$. We do not reproduce these blowups here since they are given in [SIL 1].

Suppose the reduction type of the main type is T. The presence of some zero types in the collision means that $v_i(a_j) > 0$ for some i and j. This means that some of the Weierstrass coefficients vanish in the residue field of p. As a result the special fiber at p is either type T, or similar to type T but with some components identified.

In the following subsections we discuss which components of T may be identified in the special fiber at p. We denote the residue field of O_p by κ.

16.7.1 The A6 (0,1) Collision

In this subsection we assume that the modulus of the collision is 6, and that $v_t(a_6) = 1$. In this case we also know $v_{r_i}(a_6) = 0$ for the other uniformizing parameters r_i, and that a_6 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 270 describes what the main types and possible zero types in the collision can be based on the residue characteristic of O_p.

$Char(\kappa)$	$Main Type$	$Zero Types$
3	II	$Z1$
2	II	$X1, Y1$

(270)

The special fiber is a nodal cubic. Because $\frac{a_6}{t}$ is a unit in O_p, the scheme is regular at the cusp. The extra vanishing of the other coefficients a_i has no effect on this computation. Thus the special fiber is still a type II.

16.7.2 The A6 (0,2) Collision

In this subsection we assume that the modulus of the collision is 6, and that $v_t(a_6) = 2$. In this case we also know $v_{r_i}(a_6) = 0$ for the other uniformizing parameters r_i, and that a_6 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 271 describes what the main types and possible zero types in the collision can be based on the residue characteristic of O_p.

$Char(\kappa)$	$Main Type$	$Zero Types$
3	IV	$Z1$
2	$IV, Y2$	$X1, Y1$

(271)

Following the sequences of blow ups in Tate's algorithm for type IV or $Y2$, the fact that $\frac{a_6}{t^2}$ is a unit in O_p shows that every point of X'' on the fiber above p is regular. This special fiber will be of type IV or $Y2$ depending on the number of roots of the polynomial 8 in κ.

If $char(\kappa) = 3$, the polynomial has distinct roots, but if $char(\kappa) = 2$, then $v_r(a_3) > 0$ for some index r, thus polynomial a double root in κ.

We conclude the special fiber is type IV if $char(\kappa) = 3$, and type $Y2$ if $char(\kappa) = 2$.

16.7.3 The A6 (0,3) Collision

In this subsection we assume that the modulus of the collision is 2 or 6, and that $v_t(a_6) = 3$. In this case we also know $v_{r_i}(a_6) = 0$ for the other uniformizing parameters r_i, and that a_6 is a $\{t_i\}$ normal element. If the modulus of the collision is 2, we also know that $v_t(a_2) = 1$, and that a_2 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 271 describes what the main types and possible zero types in the collision can be based on the modulus of the collision and residue characteristic of O_p.

$Char(\kappa)$	$Modulus$	$Main\,Type$	$Zero\,Types$
3	6	$I_0^*, Z2$	$Z1$
2	6	$I_0^*, X2$	$X1$
2	6	I_0^*	$Y1$
2	2	$I_0^*, X2$	$X1$

(272)

Following the sequences of blow ups in Tate's algorithm for type I_0^* or $X2$, the fact that 9 has no κ rational multiple roots shows that every point of X'' above p is regular. Given the fact that the only main type in the collision has straight discriminant 6, and rational multiple root of 9 would violate the pre-settled hypothesis. This special fiber will be of type I_0^* or $X2$ depending on the types of roots of the polynomial 9 in κ.

If $char(\kappa) = 3$, the polynomial has a triple non rational root. If $char(\kappa) = 2$, and the main type is $X2$ the polynomial has a double non rational root. If $char(\kappa) = 2$, and the zero type is $Y2$ the polynomial must have distinct roots, otherwise we can contradict the settled hypothesis. For the collisions involving I_0^* and $X1$, the result depends on the exact from of the coefficients. If $v_t(a_6 - a_4 a_2) > 0$ the polynomial the polynomial has a double non rational root. if the $char(\kappa_t) = 0$, the converse holds. It is possible to reduce to this situation, but let us be content with the following.

The special fiber is type $Z2$ if $char(\kappa) = 3$, and if $char(\kappa) = 2$ and the main type is $X2$, then the special fiber is $X2$. If $char(\kappa) = 2$ and the main type is I_0^*, then the special fiber is $X2$ or I_0^*.

16.7.4 The A6 (0,4) Collision

In this subsection we assume that the modulus of the collision is 6, and that $v_t(a_6) = 4$. In this case we also know $v_{r_i}(a_6) = 0$ for the other uniformizing parameters r_i, and that a_4 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 273 describes what the main types and possible zero types in the collision can be based on the residue characteristic of O_p.

$Char(\kappa)$	$Main Type$	$Zero Types$
3	IV^*	$Z1$
2	$IV^*, Y3$	$X1, Y1$

(273)

Following the sequences of blow ups in Tate's algorithm for type IV^* or $Y3$, the fact that $\frac{a_6}{t^4}$ is a unit in O_p shows that every point of X'' on the fiber above p is regular. This special fiber will be of type IV^* or $Y3$ depending on the number of roots of the polynomial 10 in κ.

If $char(\kappa) = 3$, the polynomial has distinct roots, but if $char(\kappa) = 2$, then $v_r(a_3) > 0$ for some index r, thus polynomial a double root in κ.

We conclude the special fiber is type IV^* if $char(\kappa) = 3$, and type $Y3$ if $char(\kappa) = 2$.

16.7.5 The A6 (0,5) Collision

In this subsection we assume that the modulus of the collision is 6, and that $v_t(a_6) = 5$. In this case we also know $v_{r_i}(a_6) = 0$ for the other uniformizing parameters r_i, and that a_5 is a $\{t_i\}$ normal element.

This case includes several types of collisions. We describe which collisions these calculations cover with a chart. Chart 274 describes what the main types and possible zero types in the collision can be based on the residue characteristic of O_p.

$Char(\kappa)$	$Main\,Type$	$Zero\,Types$
3	II^*	$Z1$
2	II^*	$X1, Y1$

(274)

Following the sequences of blow ups in Tate's algorithm for type $II*$, the fact that $\frac{a_6}{t^5}$ is a unit in O_p shows that every point of X'' on the fiber above p is regular. This special fiber will be of type always be of type II^*.

The extra vanishing of the other coefficients a_i has no effect on this computation. Thus the special fiber is still a type II^*.

16.7.6 The A4 (0,1) Collision

In this subsection we assume that the modulus of the collision is 4, and that $v_t(a_4) = 1$. In this case we also know $v_{r_i}(a_4) = 0$ for the other uniformizing parameters r_i, and that a_4 is a $\{t_i\}$ normal element.

This case includes only collisions at points of residue characteristic 2 between a main type III, and a zero type $X1$.

The special fiber is always a type III. Because $\frac{a_4}{t}$ is a unit in O_p, the scheme is regular at the point of tangency. The extra vanishing of the other coefficients a_i has no effect on this computation. Thus the special fiber is still a type III.

16.7.7 The A4 (0,2) Collision

In this subsection we assume that the modulus of the collision is 4, and that $v_t(a_4) = 2$. In this case we also know $v_{r_i}(a_4) = 0$ for the other uniformizing parameters r_i, and that a_4 is a $\{t_i\}$ normal element.

This case includes only collisions at points of residue characteristic 2 between a main type I_0^*, or $X2$ and a zero type $X1$.

As in section 16.7.3, the special fiber is a-priori a type I_0^*, or $X2$, depending on the types of roots of the polynomial 9 in κ. The fact that 9 has no κ rational multiple roots also shows that every point of X'' above p is regular.

By consulting the value of ψ in chart 201, we see that $v_t(a_2) > 1$, and $v_t(a_6) > 3$. Thus the polynomial 9 must have a double non rational root.

Thus the special fiber is still always a type $X2$.

16.7.8 The A4 (0,3) Collision

In this subsection we assume that the modulus of the collision is 4, and that $v_t(a_4) = 3$. In this case we also know $v_{r_i}(a_4) = 0$ for the other uniformizing parameters r_i, and that a_4 is a $\{t_i\}$ normal element.

This case includes only collisions at points of residue characteristic 2 between a main type III^*, and a zero type $X1$.

The special fiber is always a type III. Because $\frac{a_4}{t^3}$ is a unit in O_p, the scheme is regular at the point of tangency. The extra vanishing of the other coefficients a_i has no effect on this computation. Thus the special fiber is still a type III^*.

16.7.9 Summary of Single Collision Fibers

We summarize what components of the main type may be identified in the special fiber in a proposition and a chart.

Proposition 16.5 (Single Special Fibers)
Let X'' be the regular model of a limited Weierstrass elliptic scheme over a base B constructed by algorithm 16.3. Let $p \epsilon B$, and let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters at p. Suppose the reduction type T_1 is a main type and all other types are zero types. Then the special fiber of X'' at p is equal to T_1, or is T_1 with some components identified.

The only collisions of this type that can occur appear on chart 275, and for each type of collision, the chart specifies the special fiber.

$Main\,Type$	$Zero\,Type$	$Special\,Fiber$
II	$X1, Y1, Z1$	II
III	$X1, Y1$	III
IV	$X1, Y1$	$Y2$
$Y2$	$X1, Y1$	$Y2$
I_0^*	$X1, Y1$	$I_0^*, X2$
I_0^*	$Z1$	$Z2$
$X2$	$X1$	$X2$
IV^*	$X1, Y1$	$Y3$
$Y3$	$X1, Y1$	$Y3$
III^*	$X1, Y1$	III^*
II^*	$X1, Y1, Z1$	II^*

(275)

16.8 Multiple Type Collisions

Here we consider collisions among types in the infinite families of reduction types. These collisions involve at most one main type and at least one zero types. These collisions are described in propositions 14.5 and 14.7.

Such collisions will be described by a Weierstrass equation in multiple chart form with coefficients in the local ring O_p. If there is a main type in the collision we let $(s, t_1...t_k)$ be a discriminant compatible set of uniformizing parameters at O_p such that $s = 0$ defines this main type. Otherwise let $(t_1...t_k)$ be a discriminant compatible set of uniformizing parameters at O_p

As specified in propositions 14.5 and 14.7 the collision must be between reduction types in the a_1 or a_2 groups, and furthermore either a_1 or a_2 must be a $\{t_i\}$ normal element.

Because many types of collisions require the same blow up computations we use the most general Weierstrass equations possible and state which collisions the computation corresponds to.

16.8.1 Char $\neq 2$ Collisions

Here we discuss resolution of X over a point $p \epsilon B$ with residue characteristic not 2. By referring to the charts of section 4, we see that the new types in the infinite families, types K_n, K'_n, and T_n arise only when the residue characteristic is 2. Thus the only reduction types involved in a collision at p are of type I_n or I_n^*. These collisions are also described in 14.5.

We may use the simpler Weierstrass equation 1 to define the scheme at p, and we have also limited ourselves to collisions between types I_n and I_n^* with at most one I_n^* type and at most one I_n type with n-odd.

The special fibers will not be computed here because they follow as special cases of the arguments in the following sections. However, these sections attempt to focus on the more difficult case where the characteristic of the residue field is 2. We summarize the results of the computations in this simpler case first.

$$\begin{array}{lcc} \underline{Types} & & \underline{Central\,Fiber} \\ I_n + I_m & & I_{n+m} \\ I_n + I_m^* & (n-odd) & I^+_{(n-1)/2+m} \\ I_n + I_m^* & (n-even) & I^*_{n/2+m} \end{array} \qquad (276)$$

The I_k^+ types are described in section 16.1.

We now consider collisions over points with O_p having residue characteristic 2. The collisions may involving the new types K_n, K'_n, or T_n.

In these cases either a_2 or a_1 has normal crossings, and as in 14.7 we reduce to the case where there is at most one main type. In the same section we reduced to the case that among I_n, K_n, and K'_n types in a collision, at most one n is odd. We further assume that the local rational maps ϕ_1 or ϕ_2 are well defined. We use this morphism to show that the final coordinate patch in a series of blow ups contains no singular points.

16.8.2 The a_1 Collisions

In this section we discuss the resolution over a point p with modulus of collision 1. This means that a_1 is a unit in O_p and the only reduction types involved in the collision are I_n types.

As mentioned in section 14.2 we have reduced to the case of collisions between I_n types with at most one n odd.

It is straightforward to check that if k I_{n_k} types collide, the resulting special fiber will be of type I_N with $N = \sum n_k$.

These computations will not be reproduced because they are very similar to the computation of section 16.8.3

To perform the analysis in full detail one would use the the I_n detail charts 4.3, and the morphism ψ_1. We mention that the number of blow ups needed for each component of the discriminant divisor is half the valuation of the discriminant, rounded down. The order of the blow ups is not important for purposes of the construction of a regular model. Finally, the fact that ψ_1 is a morphism guarantees that one of a_3, a_4, or a_6 becomes a unit after the final blow up. This proves regularity for all points in the special fiber.

16.8.3 The I_n, K_n Collisions

In this section we assume that the modulus of the collision is 2, and that there are no main types involved in the collision. We assume a_2 is a unit in O_p, and that $v_i(a_1) > 1$ for all $\{t_i\}$ in the discriminant divisor. We also assume that at least one of a_3, a_4, and a_6 are $\{t_i\}$ normal elements, depending on the value of $\phi_2(p)$.

This case includes several general types of collisions. We describe these categories in a list.

1. All I_n types.
2. At least one K_n or K'_n type, one odd type.
3. At least one K_n type, no K'_n type, no odd types.
4. At least one K'_n type, no odd types.

We group these cases together, since the Weierstrass equation has a similar form in each case, and the same blow ups are required in each case. Although the same blow ups suffice to investigate the first case, we assume that there is at least one K_n, or K'_n type.

We note also that the DVR, R_i, corresponding to any I_n type must have residue characteristic 0, since otherwise we would have $v_i(a_1) = 0$. We further assume that among the I_n, K_n, or K'_n types present in a collision at most one of the n's is odd.

We group these cases together, since the Weierstrass equation has a similar form in each case, and the same blow ups are required in each case. Let $\{t_i\}$ be a set of uniformizing set of parameters for O_p such that the first few

$t_i = 0$ define the discriminant locus. Regardless of the reduction type of E_i, we begin with a subscheme of $O_p[x, y]$ defined by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{277}$$

such that for each $\{t_i\}$

	a_1	a_2	a_3	a_4	a_6
v_{t_i}	≥ 1	0	$\geq \frac{n_i}{2}$	$\geq \frac{n_i}{2}$	$\geq n_i$

(278)

We are going to perform the blow ups prescribed by Tate's algorithm for each of the reduction types over $t_i = 0$, and compute the special fiber over the closed point of O_p. To do this we set all $s_i = 0$.

The beginning special fiber is a rational curve of multiplicity 1. Tate's algorithm specifies $\frac{n_i}{2}$ blow ups for each component $t_i = 0$. In order for there to be a collision we have at least one $n_i \geq 2$, so let the first blow up be at the ideal (x, y, t_i).

For the third coordinate patch put $x = x_1t_i$ and $y = y_1t_i$. This patch is the affine subscheme of $O_p[x_1, y_1]$ defined by

$$y_1^2 + a_1x_1y_1 + \frac{a_3}{t_i}y_1 = x_1^3t_i + a_2x_1^2 + \frac{a_4}{t_i}x_1 + \frac{a_6}{t_i^2}. \tag{279}$$

The special fiber $t_i = 0$ consists of a multiplicity 2 rational curve defined by $y_1^2 = a_2x_1^2$.

This component is not defined over the residue field, but in a degree two extension of it. This is because a_2 is not a square in κ the residue field of O_p. A priori we only knew that a_2 was not a square in the residue field of the DVR corresponding to a K_n type, but the pre-settled hypothesis guarantees that it is also non square in κ.

Now repeat that blow up for a total of $\frac{n_i}{2}$ times for each component $t_i = 0$. That is blow up at an ideal (x', y', t_i), $\frac{n_i}{2}$ times for each i, after appropriately renaming the x, and y coordinates. Each such blow up produces a new multiplicity 2 component not rational over κ, except perhaps the last.

Lets now examine the last coordinate patch. Define

$$T = \Pi t_i^{[\frac{n_i}{2}]}. \tag{280}$$

Here the brackets denote the greatest integer function. As in the first blow up, each successive blow up reduces the powers of t_i in a_3, a_4, a_6, and adds powers of t_i to the x^3 coefficient. Thus it is easy to see that the last coordinate patch is given by

$$y_1^2 + a_1 x_1 y_1 + \frac{a_3}{T} y_1 = x_1^3 T + a_2 x_1^2 + \frac{a_4}{T} x_1 + \frac{a_6}{T^2}. \tag{281}$$

The special fiber is computed by setting all the $t_i = 0$. The special fiber is then

$$y_1^2 + \frac{a_3}{T} y_1 = +a_2 x_1^2 + \frac{a_4}{T} x_1 + \frac{a_6}{T^2}. \tag{282}$$

As in the analysis of the new types K_n, we are going to examine b_8, which is defined with equation 15. It is also the discriminant of quadrics such as 281.

Supposing $\phi_2 \neq (0, 0, *)$, there are no I_n or K_n with n odd and no K'_n reduction types in the collision. Then for each t_i in the discriminant locus $v_{t_i}(b_8) = n_i$, and for t_i not in the discriminant locus $v_{t_i}(b_8) = 0$. One can check that the assumption of being settled also forces b_8 to have normal crossings, so that $\frac{b_8}{T^2}$ is a unit. We check this using the phi_2 morphism.

In this case the quadric 281 is non degenerate, and the last special fiber is a multiplicity 1 component. Since the special fiber is regular, so is the scheme at points on the special fiber. The total special fiber in this case is a chain of components: one of multiplicity 1, N of multiplicity 2, and one last one of multiplicity 1. Here

$$N = \sum \frac{n_i}{2} \tag{283}$$

Now suppose $\phi_2 = (0, 0, *)$. This means for each i $v_{t_i}(a_6) = n_i$. Further suppose that there is at least one odd n_i in the collision. Examining the valuation of a_3,a_4, and a_6 for this t_i, we find that in the residue field κ

$$\frac{a_3}{T} = \frac{a_4}{T} = \frac{a_6}{T^2} = 0. \tag{284}$$

So the special fiber is the double line $y_1^2 = +a_2 x_1^2$. The settled assumption guarantees that a_2 is not a square in the residue field, so the component is not defined over κ. Furthermore it has only one rational point $x = y = 0$, and this point is not a singular point of the scheme.

Thus every point of X'' over p in this patch is regular, and we have computed the entire special fiber. Indeed every point of X'' over p is regular.

The total special fiber in this case is a chain of components: one of multiplicity 1, and N of multiplicity 2. The number of multiplicity two components is given by

$$N = \frac{\sum n_i - 1}{2}. \tag{285}$$

Now suppose still that $\phi_2 = (0, 0, *)$, but there is no odd n_i in the collision. Thus the collision consists entirely of I_n and K_n' types and each n is even. We are assuming there is at least one K_n' type, otherwise we would be in the pure I_n collision situation. For the t_i corresponding to the K_n' type, we have $v(a_3) > \frac{n}{2}$, and $v(a_4) > \frac{n}{2}$. So the special fiber is the double line

$$y_1^2 = +a_2 x_1^2 + \frac{a_6}{T^2}. \tag{286}$$

The settled assumption guarantees that a_2 is not a square in the residue field and also that the double line has no rational points at all. As in the analysis of the K_n' type, the special fiber is regular, and thus so is the scheme at points on the special fiber. The total special fiber in this case is a chain of components: one of multiplicity 1, and N of multiplicity 2. The number of multiplicity two components is given by

$$N = \sum \frac{n_i}{2}. \tag{287}$$

To summarize the special fiber in a collision of types K_n, K_n', and I_n types is again a configuration of components of one of the K_n, K_n', and I_n types. If there are no I_n or K_n with n odd and no K_n' reduction types in the collision the special fiber is of type K_m for some even m. If there is one I_n or K_n with n odd, the special fiber is of type K_m for some odd m. If there are no I_n or K_n with n odd, but one K_n' type the special fiber is of type K_m'.

16.8.4 The T_n or I_n^* (n odd) Collisions

In this section we assume that the modulus of the collision is 2, and that there is one main type involved in the collision. We assume that this main type I_n^* or T_n has n odd. As above, we note that the DVR corresponding to such an I_n type must have residue characteristic 0. We can further assume that among the I_n, K_n, or K_n' types present in a collision at most one of the n's is odd.

We also assume that at least one of a_3, a_4, and a_6 are $\{t_i\}$ normal elements, depending on the value of $\phi_2(p)$.

This case includes several general types of collisions. We describe these categories in a list.

1. At least one I_n or K_n odd type.
2. The main type is T_n, no odd types.
3. The main type is I_n^*, at least one K_n' type, no odd types.
4. The main type is I_n^*, no K_n' type, no odd types.

We group these cases together, since the Weierstrass equation has a similar form in each case, and the same blow ups are required in each case.

In the computations that follow, let s be the uniformizing parameter for O_p such that $s = 0$ defines the I_n^* or T_n type. Let $\{t_i\}$ be the uniformizing parameters that define the I_n, K_n, or K_n' types.

We begin with the standard subscheme of $O_p[x, y]$ defined by a Weierstrass equation.

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{288}$$

such that for s and each t_i

	a_1	a_2	a_3	a_4	a_6
v_s	≥ 1	1	$\geq \frac{n+3}{2}$	$\geq \frac{n+5}{2}$	$\geq n+3$
v_{t_i}	≥ 1	0	$\geq \frac{n_i}{2}$	$\geq \frac{n_i}{2}$	$\geq n_i$

(289)

We are going to perform the blow ups prescribed by Tate's algorithm for each of the reduction types over $s = 0$ and $t_i = 0$, and compute the special fiber over the closed point of O_p. To do this we set all $s = t_i = 0$.

The beginning special fiber is a rational curve of multiplicity 1. The first blow ups will be the same as those described in the T_i new type, even if the main type is a I_n^* type. We will perform these first, then the $\frac{n_i}{2}$ blow ups for each component $t_i = 0$. Briefly recalling the sequence of blow ups for the T_i new type, we first blow up at the ideal point (x, y, π). For the third coordinate patch put $x = x_1 s$ and $y = y_1 s$. This patch is the affine subscheme of $O_p[x_1, y_1]$ defined by

$$y_1^2 + a_1 x_1 y_1 + \frac{a_3}{s} y_1 = x_1^3 s + a_2 x_1^2 + \frac{a_4}{s} x_1 + \frac{a_6}{s^2}. \tag{290}$$

The special fiber consists of a multiplicity 2 rational curve defined by $y_1^2 = 0$

Next blow up at the double line (y_1, s). For the second coordinate patch put $y_1 = y_2 s$. This patch is the affine subscheme of $O_p[x_1, y_2]$ defined by

$$y_2^2 s + a_1 x_1 y_2 + \frac{a_3}{s} y_2 = x_1^3 + \frac{a_2}{s} x_1^2 + \frac{a_4}{s^2} x_1 + \frac{a_6}{s^3}. \tag{291}$$

The special fiber is $x_1^3 + \frac{a_2}{s} x_1^2 = 0$. This consists of the multiplicity 1 rational curve $x + \frac{a_2}{s} = 0$, and the double line $x_1^2 = 0$.

Next blow up at the double line (x_1, s). For the second coordinate patch put $x_1 = x_2 s$. This patch is the affine subscheme of $O_p[x_2, y_2]$ defined by

$$y_2^2 + a_1 x_2 y_2 + \frac{a_3}{s^2} y_2 = x_2^3 s^2 + a_2 x_2^2 + \frac{a_4}{s^2} x_2 + \frac{a_6}{s^4}. \tag{292}$$

The special fiber is $y_2^2 = 0$. since there are powers of t_i in the a_i coefficients.

Supposing $n = 1$, we stop. Otherwise repeat the last two blow ups. The pair of blow ups is performed a total of $\frac{n+1}{2}$ times. So far including the last coordinate patch the special fiber is two multiplicity one rational curves connected to a chain of $n+1$ multiplicity 2 curves. The last coordinate patch has the form

$$y'^2 + a_1 x'y' + \frac{a_3}{s^{\frac{n+3}{2}}} y' = x'^3 s^{\frac{n+3}{2}} + a_2 x'^2 + \frac{a_4}{s^{\frac{n+3}{2}}} x' + \frac{a_6}{s^{n+3}}. \tag{293}$$

As in the previous subsection 16.8.3, we blow up successively at ideals such as (x', y', t_i). We repeat these types of blow up for a total of $\frac{n_i}{2}$ times for each component $t_i = 0$. After each blow up, we examine the third coordinate patch. Each such blow up produces a new multiplicity 2 component in the special fiber, except perhaps the last.

We now now examine the last coordinate patch. Define

$$T = \Pi\, t_i^{[\frac{n_i}{2}]}. \tag{294}$$

Here the brackets denote the greatest integer function. As in the first blow up, each successive blow up reduces the powers of t_i in a_3, a_4, a_6, and adds powers of t_i to the x^3 coefficient. Thus it is easy to see that the last coordinate patch is given by

$$y'^2 + a_1 x'y' + \frac{a_3}{s^{\frac{n+3}{2}} T} y' = x'^3 s^{\frac{n+3}{2}} T + a_2 x'^2 + \frac{a_4}{s^{\frac{n+3}{2}} T} x' + \frac{a_6}{s^{n+3} T^2}. \tag{295}$$

The special fiber is computed by setting all the $s = t_i = 0$. Because of the powers of s in a_1, and a_2, and the powers of t_i in a_4, these terms drop out and the special fiber is of the form

$$y'^2 + \frac{a_3}{s^{\frac{n+3}{2}} T} y' = \frac{a_6}{s^{n+3} T^2}. \tag{296}$$

We now determine exactly what this last addition to the special fiber is.

Suppose there is one K_n, or I_n with n odd. The additional power of that t_i in a_3 and a_6 makes the special fiber the double line $y'^2 = 0$. Furthermore this can only happen when $\phi_2 = (0, 0, *)$. This means that $\frac{a_6}{s^{n+3} T^2}$ is t_i times a unit in O_p, and that the patch defined by 295 is regular.

Now suppose there are no K_n, or I_n types with n odd, but there is a T_n type. Then the additional power of s in a_3 makes the special fiber the double line $y'^2 = \frac{a_6}{s^{n+3}T^2}$. Furthermore the settled hypothesis guarantees that $\frac{a_6}{s^{n+3}T^2}$ is not a square in the residue field κ. So this last multiplicity 2 component is not rational over κ. Since the double line has no rational points, the scheme is regular at these points.

Also if there is a K'_n involved in the collision there is an additional power of t_i in a_3, and the addition to the special fiber is also a non rational multiplicity 2 component.

Otherwise, there is no T_n, or K'_n, or K_n with n odd, so examining the ϕ_2 chart 187 we see that we must have $\phi_2 = (*, 0, 0)$ or $\phi_2 = (0, *, 0)$, because there must be an I^*_n type with n odd present. For either form of ϕ_2 we know that a_3 is a $\{t_i\}$ normal element and in fact that $\frac{a_3}{s^{\frac{n+3}{2}}T}$ is a unit in O_p. This means that the quadratic 295 has distinct roots and the addition to the special fiber is also two a multiplicity 1 components. Since these components have multiplicity 1, the scheme is regular in this last coordinate patch.

To summarize, the special fiber is of the form T_n, I^*_n, or I^+_n. If there is a K_n, or I_n with n odd, we have special fiber type I^+_n. If there is a T_i, or K'_n type, we have special fiber type T_n. Otherwise, there are only I^*_n, and I_n with n even and K_n with n even in the collision, and then we have special fiber type I^*_n.

To count the number of multiplicity 2 components, set

$$N = n + \frac{\sum n_i}{2}. \tag{297}$$

The number of multiplicity 2 components in the I^+_n or I^*_n types is $N+1$. The number of multiplicity 2 components in the T_n type is N.

16.8.5 The T_n or I^*_n (n even) Collisions

In this section we assume that the modulus of the collision is 2, and that there is one main type involved in the collision. We assume that this main type I_n^* or T_n has n odd. This section covers the exact same cases as section 16.8.4, except that the type I_n^* or T_n has n odd.

Thus we perform blow ups to resolve the elliptic scheme over a point p with where one I_n^* or T_n type with n odd and 1 or more I_n,K_n, or K_n' types collide.

The only slight difference from section 16.8.4 is that one further double line blow up is required for the I_n^* or T_n type.

After following the blow ups of the previous section, we obtain the coordinate patch

$$y'^2 s + a_1 x' y' + \frac{a_3}{s^{\frac{n+2}{2}}} y' = x'^3 s^{\frac{n}{2}} + \frac{a_2}{s} x'^2 + \frac{a_4}{s^{\frac{n+2}{2}}} x' + \frac{a_6}{s^{n+3}}. \tag{298}$$

The special fiber is the double line $\frac{a_2}{s} x'^2 = 0$. At this point there are $n+1$ double lines.

We then blow up as above at ideals of the form (x, y, t_i). Each blow up also produces another multiplicity component. In this case the last coordinate patch is

$$y'^2 s + a_1 x' y' + \frac{a_3}{s^{\frac{n+2}{2}}} y' = x'^3 s^{\frac{n}{2}} T + \frac{a_2}{s} x'^2 + \frac{a_4}{s^{\frac{n+2}{2}} T} x' + \frac{a_6}{s^{n+3} T^2}. \tag{299}$$

where T is defined as above in 280. In this case $a_1 = a_3 = 0$ in the special fiber, so the part of the special fiber in this coordinate patch is

$$0 = \frac{a_2}{s} x'^2 + \frac{a_4}{s^{\frac{n+2}{2}} T} x' + \frac{a_6}{s^{n+3} T^2}. \tag{300}$$

As above we determine the special fiber in this last patch by looking at whether or not there is an I_n, or K_n with n odd, then whether or not there is a T_n or K_n' present in the collision. The results are the same as the previous case, and only the computation was slightly different.

Thus each point of X'' above p is a regular point, and the special fibers type were given in section 16.8.4. The formulas for the number of multiplicity 2 components given in section 16.8.4 also hold in this n even case.

16.8.6 Summary of Multiple Collision Fibers

Here we summarize the collisions involving reduction types in one of the infinite families. These collisions may involve the standard I_n or I_n^* types, or some of the new types K_n,K_n', or T_n.

If all reduction types in the collision are of type I_n, we know that the special fiber is also of type I_n. Otherwise, we describe the special fiber at p based on three properties. The first is: does the collision contains a I_n^*, T_n, or no main type? Second, does it contains a K_n' type? Third, do any of the I_n, or K_n types have n odd?

Proposition 16.6 (Multiple Special Fibers)
Let X'' be the regular model of a limited Weierstrass elliptic scheme over a base B, constructed by algorithm 16.3. Let $p \epsilon B$, and let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters at p. Suppose the reduction types T_1 and T_2 are main type and all other types are zero types.

Let X'' be the regular model of a limited Weierstrass elliptic scheme over a base B. Let $p \epsilon B$, and let $\{t_i\}$ be a discriminant compatible set of uniformizing parameters at p. Suppose the modulus of the collision is 1 or 2, and that the reduction types involved in the collision are types I_n, I_n^*, K_n, K_n', and T_n.

If all reduction types are I_{n_i}, then the special fiber is also of type I_N with

$$N = \sum n_i \tag{301}$$

Otherwise the reduction type is one of I_n^*, I_n^+, K_n, K_n', or T_n. The special fiber only depends three pieces of data

1. *What main type, if any is in the collision.*
2. *Whether or not a K_m' type is in the collision.*
3. *Whether or not a I_n or K_n type with n odd is in the collision.*

Chart 303 specifies the special fiber based on these criteria. The number of multiplicity 2 components in the special fiber is specified in last column, as a *function of* N*, where*

$$N = n + \sum [\frac{n_i}{2}] \tag{302}$$

where n is the subscript of the main type I_n^* *or* T_n *and the* n_i *are the subscripts of the zero types* I_n*,* K_n*, or* K_n'*.*

Main Types	K_m' *Types*	*Odd n*	*Special Fiber*	$2 - $ *Components*
$-$	$*$	*yes*	K_m	N
$-$	*no*	*no*	K_m	$N-1$
$-$	*yes*	*no*	K_m'	N
I_n^*	$*$	*yes*	I_m^+	$N+2$
T_n	$*$	*yes*	I_m^+	$N+2$
T_n	$*$	*no*	T_m	$N+1$
I_n^*	*yes*	*no*	T_m	$N+1$
I_n^*	*no,*	*no*	I_m^*	$N+1$

(303)

16.8.7 Collision Summary

By the compuations of sections 16.6, 16.7, and 16.8, every point of the scheme $X'' \to B$ constructed in section 16.3.1 is regular. This proves theorem 16.2, which states that algorithm 16.3 does suffice to desingularize a limited Weierstrass elliptic scheme. This concludes the main theorem 1.2 of the paper.

In section 17 we will assemble the additional information about the fibers of the regular model.

17 Special Fibers

In this section we strengthen theorem 1.2 by describing explicitly the fibers over each point of the base. First let us give a name to such a model.

Definition 17.1 *Let B be a regular Noetherian n - dimensional integral separated scheme. Let $X \to B$ be a Weierstrass elliptic scheme over B. Suppose there exists a blow up $B' \to B$ defining the base change*

$$\begin{array}{ccccc} X_{min} & \leftrightarrow & X' & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ B' & = & B' & \to & B \end{array}$$

and a minimal Weierstrass elliptic scheme X_{min} birational to X' over B'.

Suppose that $X_{min} \to B'$ is a limited Weierstrass elliptic scheme, and that $X'' \to X_{min}$ is the series of blow ups specified by algorithm 16.3. Then $X'' \to B'$ is a Flat Collision Model of $X \to B$.

Corollary 17.2 (Special Fiber Possibilities)
Let B be a regular Noetherian n - dimensional integral separated scheme. Let $X \to B$ be a Weierstrass elliptic scheme over B. Suppose $X'' \to B'$ is a flat collision model for $X \to B$.

The special fibers of X'' are nonsingular elliptic curves, reduction types on Kodaira's list, or are one of the following types defined in section 6.

$$X1, X2, Y1, Y2, Y3, Z1, Z2, K_n, K_n', T_n, \tag{304}$$

the collision type

$$I_m^+ \tag{305}$$

or a chains of rational curves with multiplicities

$$1-2-3 \tag{306}$$

$$1-2-3-4-2 \tag{307}$$

$$1-2-3-2 \tag{308}$$

$$1-2-3-2-1 \tag{309}$$

Corollary 17.3 (Recognize Fiber Types)
Let B be a regular Noetherian n - dimensional integral separated scheme. Let $X \to B$ be a Weierstrass elliptic scheme over B. Suppose $X'' \to B'$ is a flat collision model for $X \to B$. The fiber types in corollary 17.2 have all appeared as either special fibers in a flat collision model over bases with no points or residue characteristic 2 or 3, or as special fibers in a flat collision model over a DVR of residue characteristic 2 or 3.

Furthermore the special fiber at a collision point p in the base is determined by the reduction types involved in the collision. In the following corollary we specify a collision type by what reduction types are involved.

Corollary 17.4 *Let B be a regular Noetherian n - dimensional integral separated scheme. Let $X \to B$ be a Weierstrass elliptic scheme over B. Suppose $X'' \to B'$ is a flat collision model for $X \to B$.*

Then X'' has only collisions of type specified in chart 310. Suppose $p \epsilon B'$ is a collision point of the elliptic scheme.

The special fiber of $X'' \to B'$ at p is is one of the types listed in corollary 17.2, and it is determined by the reduction types of each component of the discriminant divisor passing through p as specified in 310.

Further details concerning the various types of collisions may be found in sections 16.6, 16.7, and 16.8.

We make some remarks about chart 310.

1. At least two types must be present in each collision.
2. All main types listed must be present.
3. Zero or more of the zero types listed may be present.
4. The notation $K'n^*$ means at least one $K'n$ type must be in the collision.

$Main\,1$	$Main\,2$	$ZeroTypes$	$Oddn$	$Char$	$Special\,Fiber$
II	I_0^*	$X1, Y1, Z1$		any	$1-2-3$
II	$X2$	$X1, Y1$		2	$1-2-3$
II	$Z2$	$Z1$		3	$1-2-3$
IV	I_0^*	$X1, Y1, Z1$		any	$1-2-3-2$
IV	$X2$	$X1, Y1$		2	$1-2-3-2$
IV	$Z2$	$Z1$		3	$1-2-3-2$
$Y2$	I_0^*	$X1, Y1$		2	$1-2-3-2$
$Y2$	$X2$	$X1, Y1$		2	$1-2-3-2$
II	IV^*	$X1, Y1, Z1$		any	$1-2-3-4-2$
II	$Y3$	$X1, Y1$		2	$1-2-3-4-2$
III	I_0^*	$X1, Y1$		any	$1-2-3-2-1$
III	$X2$	$X1$		2	$1-2-3-2-1$
$-$	$-$	I_n		any	I_n
$-$	$-$	$I_n, K_n, K'n$	yes	2	K_n
$-$	$-$	I_n, K_n	no	2	K_n
$-$	$-$	$I_n, K_n, K'n^*$	no	2	K'_n
I_n^*	$-$	$I_n, K_n, K'n$	yes	any	I_m^+
T_n	$-$	$I_n, K_n, K'n$	yes	2	I_m^+
T_n	$-$	$I_n, K_n, K'n$	no	2	T_m
I_n^*	$-$	$I_n, K_n, K'n^*$	no	2	T_m
I_n^*	$-$	I_n, K_n	no	any	I_m^*
II	$-$	$X1, Y1, Z1$		$2, 3$	II
III	$-$	$X1, Y1$		2	III
IV	$-$	$X1, Y1$		2	$Y2$
I_0^*	$-$	$X1, Y1$		2	$I_0^*, X2$
I_0^*	$-$	$Z1$		3	$Z2$
$X2$	$-$	$X1$		2	$X2$
IV^*	$-$	$X1, Y1$		2	$Y3$
$Y3$	$-$	$X1, Y1$		2	$Y3$
III^*	$-$	$X1, Y1$		2	III^*
II^*	$-$	$X1, Y1, Z1$		$2, 3$	II^*

(310)

18 Overview

In section 1 I state the goal of constructing a flat regular elliptic subscheme.

In section 2 I consider general Weierstrass equations and schemes.

In section 3 I consider extending Tate's algorithm to non perfect residue fields.

In section 4 I display charts of Weierstrass coefficient valuations and define types.

In section 5 I prove that we can translate to pre-chart form.

In section 6 I compute new special fibers by computing with blow ups.

In section 7 I summarize new reduction types geometrically.

In section 8 I define higher dimensional elliptic schemes and consider translations.

In section 9 I consider blowing up the base and computing the exceptional divisor.

In section 10 I define groups of reduction types, assume X is pre-settled, and define ϕ.

In section 11 I show that we can translate to multiple chart form.

In section 12 I assume X is settled and define ψ.

In section 13 I discuss stable properties of settled elliptic schemes.

In section 14 I reduce collisions with combinatorics.

In section 15 I show a J morphism implies pre-settled.

In section 16 I construct the regular model, check regularity and compute collision fibers.

In section 17 I summarize all special fibers computed.

19 References

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