Algorithm to Construct a Regular Flat Model of an Elliptic Schemes

Michael Szydlo

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Contents

1 Introduction

The purpose of this paper is present an algorithm to construct a flat resolution of an elliptic scheme defined over a variety of characteristic zero.

I treated this topic in further generality in my PHD. thesis, and prove that such a flat resolution exists. The thesis also proves the algorithm fro more general elliptic schemes

The goal of this paper will be to succinctly describe what blow ups are needed to construct the model.

First let us describe the elliptic schemes that we start with.

Definition 1.1 (Weierstrass Elliptic Scheme) *Suppose B is a smooth variety defined over a field of characteristic 0. Let X be a variety defined over B by Weierstrass equations. That is, for each open $U = Spec(R)$ of B, X is defined as the projectivization of the subscheme of $Spec(R[x, y])$ cut out by an equation*

$$y^2 = x^3 + a_4 x + a_6 \tag{1}$$

for some $a_4, a_6 \epsilon U$

Such an $X \to B$ is called a Weierstrass Elliptic Scheme.

To say that X has a flat resolution is the content of the following theorem.

Theorem 1.2 (Flat Resolution)
Let X be a Weierstrass elliptic scheme over B. Then there exists a blowup $B' \to B$ and scheme $X' \to B'$ birational to X such that $X' \to B'$ is regular proper and flat.

We obtain more than a resolution of X, which would be guaranteed by Hironaka's desingularization theorem. In particular, the model we construct is flat over a the base B', and we can describe all of the fibers. We also note that not finite extension of the function field is required. We further mention that this result generalizes a result of Miranda, who dealt with two dimensional bases.

The proof of the theorem as well the construction break down into 9 steps. We shall not prove each step as in the thesis, but will rather focus on the construction of the flat resolution.

1. Reduce to the case where the reduced discriminant locus has normal crossings, and that there is a morphism $J : B \to P^1$ extending the j invariant for smooth elliptic curves.

2. Assign a standard Kodaira reduction type to each component of the discriminant locus.

3. Define 3 open subschemes S_1, S_2, and S_3 of the base depending on the value of J.

4. For each component of the discriminant locus meeting an open set S_i, define an integer invariant depending on its Kodaira type.

5. Perform a series of blow ups of B according to an algorithm which specifies the closed subschemes of B of the blow ups in terms of the invariants defined above.

6. The series of blowups defines a morphism $B' \to B$. Define X_1 as the pullback of this morphism.

7. Construct a scheme $X_2 \to B'$ birational to X_1 be replacing all Weierstrass equations with minimal Weierstrass equations.

8. Order the components of the discriminant divisor, and for each component perform a series of blow ups of X_2. The subschemes of X_2 defining the blowups are higher dimensional analogs to those present in the proof of Tate's algorithm.

9. Check that the resulting Scheme is regular and flat by examining the tangent spaces. This may be done explicitly in coordinates.

I will now give details for each step of the construction. Further discussion may be found in the thesis.

2 Preliminary Reductions

The discriminant defines a divisor D in B. Blow up the base B, until the reduced preimage of D has normal crossings. The fact that this process terminates is a consequence of the general resolution of singularities theorem in characteristic zero.

To define the morphism $J : B \to P^1$, just resolves the rational map c_4^3/d. In our case c_4 is just a multiple of the a_4 above.

3 Assigning Kodaira types

This assignment is standard and may summarized as follows.

Suppose A is a component of the discriminant locus. Let $p\epsilon A$ be a point that is not contained in any other component of the discriminant locus. Suppose that A is defined locally by $t = 0$ in a neighborhood of p.

First, suppose $J(p) \neq \infty$, and let $v_t(d)$ be the highest power of t dividing the discriminant d. Then we assign a Kodaira type just by examining the valuation of the discriminant

as follows

$$\begin{array}{cc} v_t(d)\, mod\, 12 & Type \\ 0 & I_0 \\ 2 & II \\ 3 & III \\ 4 & IV \\ 6 & I_0^* \\ 8 & IV^* \\ 9 & III^* \\ 10 & II^* \end{array} \tag{2}$$

If, on the other hand, $J(p) = \infty$ and $v_t(a_6) = 0 \pmod 6$ then we assign type I_n to A. Finally, if $J(p) = \infty$ and $v_t(a_6) = 3 \pmod 6$ we assign type I_n^* to

A. In the last two cases the integer n is given by $n = v_t(d) - 3v_t(a_6)$.

In this step we merely assigned types to components of the discriminant locus. The fact that a proper regular model of an elliptic scheme with the above valuation pattern has the special fiber described by the Kodaira type is an application of Tate's algorithm.

The above summary can also be found in a chart in Silverman's second volume. The fact that a_6 or d must have the above form is an easy calculation with a_4,a_6, and d.

4 Defining Subschemes

The next step is to define three open subschemes of the base which cover B.

Define S_1 to be the open subscheme of B with $J \neq 1728$ and $J \neq \infty$.

Define S_2 to be the open subscheme of B with $J \neq 0$ and $J \neq \infty$.

Define S_3 to be the open subscheme of B with $J \neq 0$ and $J \neq 1728$.

Notice that the only Kodaira types meeting S_1 are types II, IV, I_0^*, IV^*, and II^*. Also, the only Kodaira types meeting S_2 are types III, I_0^*, and III^*, and the only Kodaira types meeting S_3 are types I_n, I_0^*, and I_n^*.

This follows from the standard computations of J.

Notice also that the Kodaira type I_0^* may meet more than one of the open subschemes S_1, S_2, and S_3.

5 Defining the Invariants

From this point on, we deal with the three open subschemes separately.

For each component of the discriminant divisor that meets an open subscheme S_i we associate an integer. This integer depends on the reduction type. Let λ denote this association. Note that this invariant is relative to the particular open set S_i.

For components in S_1 define the invariant as follows:

$$\begin{array}{cc} Type & \lambda \\ I_0 & 0 \\ II & 1 \\ IV & 2 \\ I_0^* & 3 \\ IV^* & 4 \\ II^* & 5 \end{array} \tag{3}$$

For components in S_2 define the invariant as follows:

$$\begin{array}{cc} Type & \lambda \\ I_0 & 0 \\ III & 1 \\ I_0^* & 2 \\ III^* & 3 \end{array} \tag{4}$$

For components in S_3 define the invariant as follows:

$$\begin{array}{cc} Type & \lambda \\ I_n & 0 \\ I_n^* & 1 \end{array} \tag{5}$$

6 Blowing up the base

Each blow up of the base will be entirely contained in one open set S_i, and involves a separate argument in each of the three cases. Let us first focus on S_1 because it is the most interesting case.

We use the following notation: A $(5,5)$ blow up is a blow up at the reduced subscheme of B defined by two components A and B of the discriminant locus with both $\lambda(A) = 5$ and $\lambda(B) = 5$. This is namely the intersection of two components of type II^*. Similarly, we write $(4,1,1)$ to denote a blow up at the reduced subscheme of B defined by three components A, B and C of the discriminant locus with $\lambda(A) = 4$, $\lambda(B) = 1$, and $\lambda(C) = 1$. This is namely the intersection of two components of type II, and one of type IV^*.

Before using this notation to specify a long list of blow ups to perform, we state the remarkable fact that the exceptional divisor of the blow up has a type which is uniquely determined by the types of the components involved in the blow up.

Proposition 6.1 (Lambda Calculation)
Let A and B be components of the discriminant locus meeting at a point in S_1. Let C be the exceptional divisor of the blow up defined by the intersection of A and B. Then C is also a component of the discriminant divisor, and

$$\lambda(C) = \lambda(A) + \lambda(B) \quad mod\, 6. \tag{6}$$

The proof of this fact depends on the fact that the discriminant divisor has normal crossings and thus a_6 has a very special form. As an easy but representative exercise, blow up the base $C(s,t)$ of the elliptic scheme defined by $Y^2 = x^3 + st^2$ at the ideal (s,t), and pull back the Weierstrass equation to the new base. The parameter defining the exceptional divisor will have exponent 3 in the new Weierstrass equation.

The same proposition is true if one blows up at the intersection of three or more components. This proposition allows us to calculate exactly the type of the exceptional divisor. For example the exceptional divisor of a $(5,5)$ blow up is a 4 type. Note that if there were a third component (say of type 2) meeting both 5 types at the same point, the new 4 type would meet this third 2 type in the blown up base.

It is also useful to blow up the base at intersections of three components. Foe example, the exceptional divisor of a $(4,1,1)$ blow up is a 0, (I_0) type. By a judicious choice of blow ups, one can severely restrict the types of non zero components that meet.

Proposition 6.2 (Limiting Collisions)
The following algorithm 7 will reduce us to the case where at most two non

zero components of the discriminant locus meet, and if two non zero components do meet, they must be one of the following 3 pairs: $(1,3)$*,* $(1,4)$*, or* $(2,3)$*. In Kodaira notation, these collisions are* (II, I_0^*)*,* (II, II^*)*, or* (IV, I_0^*)*.*

The algorithm is as follows: Blow up at all collisions of the first type in the following list (7), then at all collisions of the second type, etc. After each blow up assign the appropriate lambda and kodaira type to the exceptional divisor as above.

$$\begin{array}{ccccc} (5,5) & (5,4) & (5,3) & (5,2) & (5,1) \\ (4,4) & (4,3) & (4,2) & (4,1,1) & (3,3) \\ (1,1) & (3,2,1) & (2,2,2) & (3,2,2) & (3,2,1) \\ (2,2) & (4,2) & (2,1) & & \end{array} \qquad (7)$$

To prove the proposition, one just keeps track of what new types are created at each stage and what types of collisions have been completely eliminated. In the case of S_1, only $(1,3)$, $(1,4)$, and $(2,3)$ pairs may be left.

I would like next briefly discuss the collisions involved in The S_2 and S_3 open subschemes. As above, define the λ invariant for each component of the discriminant locus. The series of blow ups is given by the following list:

$$(3,3) \quad (3,2) \quad (3,1) \quad (1,1) \quad (2,1) \qquad (8)$$

In this case only $(2,1)$ i.e. (III, I_0^*) collisions remain.

For the S_3 open subscheme, there is a slight twist, and we need a proposition.

Proposition 6.3 (Multiplicative Reduction)
Let A *and* B *be components of the discriminant locus of type* I_m *and* I_n *meeting at a point in* S_3*. Let* C *be the exceptional divisor of the blow up defined by the intersection of* A *and* B*. Then* C *is also a component of the discriminant divisor, and has type* I_{m+n}

The algorithm for S_3 is as follows: First perform $(1,1)$ blow ups. Now no two $\lambda = 1$ components meet.

Next then blow up at (I_n, I_m) intersections when both n and m are odd.

Now in any remaining collision, there is at most one I_n^* type, and at most one I_m type with m odd. We may have however, any number of I_m types with m even.

7 Pullback

The series of blow ups of the base in the previous section define a composite morphism $B' \to B$. Pull back the Scheme X via this morphism. In other words, define

$$X_1 = X \times_B B' \tag{9}$$

8 Minimal Equations

We now construct a scheme X_2 from the scheme X_1 constructed above directly from the equations defining X_1. We first remark that X_1 is still a Weierstrass elliptic scheme, and we will use the Weierstrass equation defining it. We construct X_2 locally, and later patch the schemes together.

Let U be an affine open set of B' on which X_1 is defined by $y^2 = x^3 + a_4 x + a_6$. Let $t = 0$ define a component of the discriminant locus and let k be the largest integer such that $T^{2k} | a_4$ and $T^{3k} | a_6$. Then define $a_4' = a_4/2k$ and $a_6' = a_6/3k$. Repeat this for each component of the discriminant locus in U.

Now define X_2 over U by the equation $y^2 = x^3 + a_4' x + a_6'$. Because each component of the discriminant locus has a well defined type, these schemes patch together over an open cover of B' to define a scheme $X_2 \to B'$. It is easy to see that the scheme X_2 is birational to X_1.

Note that this is nothing more than a generalization of the concept of passing to a minimal Weierstrass equation over a discrete valuation ring.

9 Desingularizing Total Space

We will now produce the regular model from the scheme X_2 constructed above. X_2 has the property that very few types of collision can occur, and it also has the minimality property of the previous section.

Essentially this step is just an application of Tate's algorithm for each component of the discriminant divisor. The fact that the below ordering is sufficient to construct a regular model is a theorem treated in the thesis. The key item to check is that the resulting scheme is regular above points in the new base which belong to more than one component of the discriminant divisor

Perform Tate's algorithm for all components in the discriminant divisor of the first item in the list, then for all components of the second type, etc. One order which works is the opposite of the order that the types are usually listed:

$$II^*, III^*, IV^*, I_0^*, IV, III, II, I_n^* \tag{10}$$

For convenience I specify here some (or all) of the blow ups in coordinates required by Tate's algorithm.

Type II requires no blow ups.

For types III and IV blow up at (x, y, t).

For types I_0^* and IV^* blow up at first at (x, y, t). Then in the third coordinate patch (I.e. $y't = y$, $x't = x$), blow up at (y', t).

For types III^* and II^*, blow up as in type IV^*, but further blow ups are needed in more than one subsequent coordinate patches.

I_1 requires no blow ups. For I_2 or I_3 blow up at $(x - \alpha, y, t)$ where α if a double root of $x^3 + a_4x + a_6$. For I_n with $n \geq 4$ more blow ups are required in the third coordinate patch.

For I_1^* first blow up as in I_0^*. Then at the double root of $x^3 + \frac{a_4}{t}x + \frac{a_4}{t^3}$. For I_n^* with $n \geq 2$ more blow ups are required in the second coordinate patch(es).

10 Checking Regularity

To prove that the model is indeed regular one can check the dimensions of all of the cotangent spaces. I prove that the model is regular in my thesis. The fact that it is flat then follows from the dimensions of the fibers.

Finally I present a list of fibers of the flat model.

Over points of B not on the discriminant divisor the fibers of the map $X'' \to B$ are non-singular elliptic curves.

Over non-singular points of the discriminant divisor the fibers of the map $X'' \to B$ are the reduction types on Kodaira's list.

The only types of collisions that occur between reduction types are as described above, and over these singular points of the discriminant divisor the fibers of the map $X'' \to B$ are also given by chart 11.

$$\begin{array}{ccc} \underline{Types\,in\,Collision} & & \underline{Special\,Fiber} \\ II + I_0^* & & 123 \\ II + IV^* & & 12342 \\ IV + I_0^* & & 1232 \\ III + I_0^* & & 12321 \\ I_n + I_m & & I_{n+m} \\ I_n + I_m^* & (n - odd) & I^+_{(n-1)/2+m} \\ I_n + I_m^* & (n - even) & I^*_{n/2+m} \end{array} \tag{11}$$

The special fibers appearing in the last column of chart 11 consist of various rational curves of given multiplicities intersecting transversally.

The I_k^+ type consists of 2 multiplicity 1 components connected to a chain of $k+2$ multiplicity 2 components. This is similar to a type ${I_k}^*$, which has $k+1$ multiplicity 2 components, but a pair of final multiplicity 1 components. Thus type I_k^+ looks like type I_k^* with the final two components identified.

In the collisions specified by one of the last three lines in chart 11 there can be any number of I_n types present in the collision, up to the dimension of the base scheme. However, there may be at most one I_n type with n odd. If there are multiple I_n types colliding at a point, the special fiber is still given by chart 11, with n replaced by $\sum n_i$.